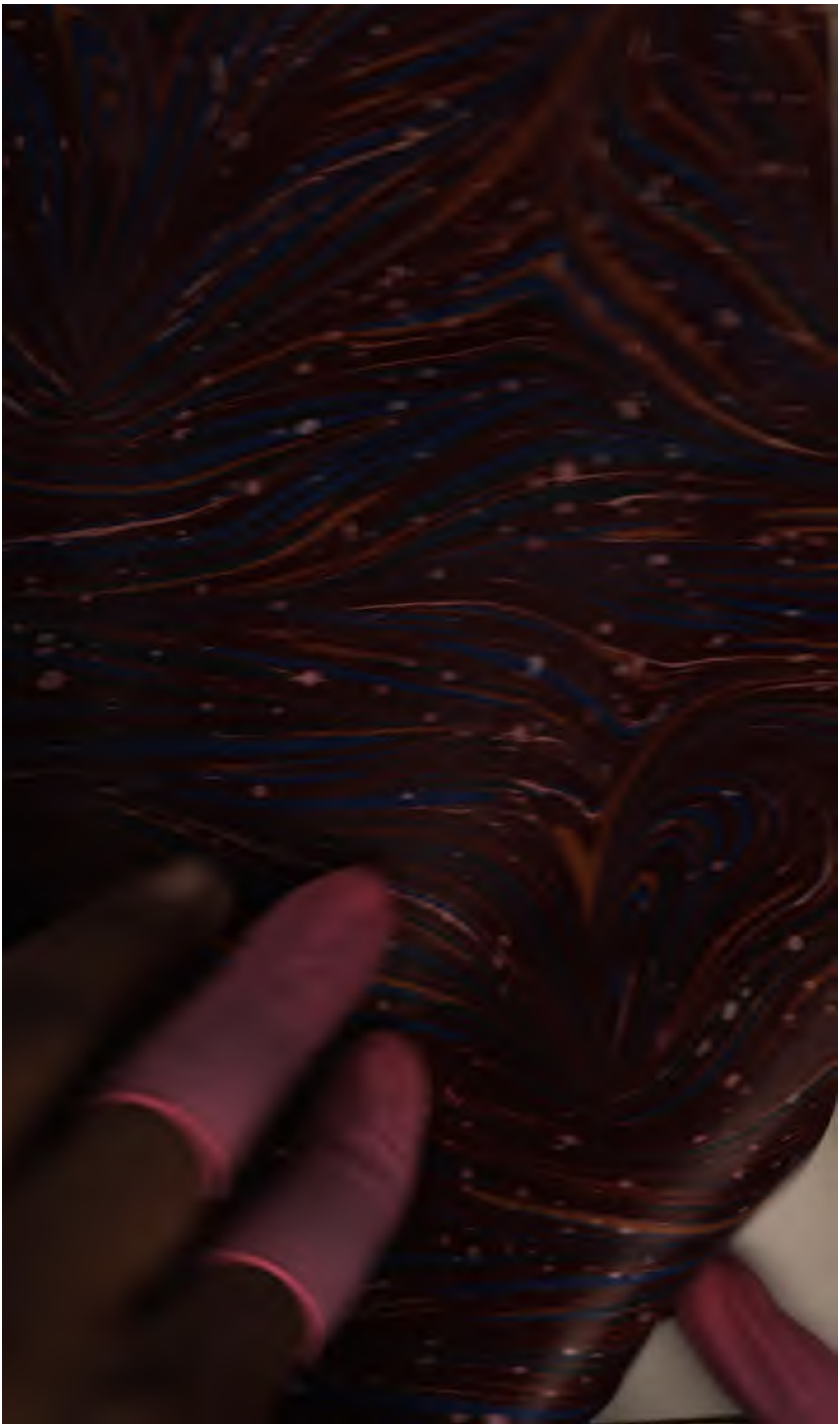

A TREATISE

ON

HYDROMECHANICS

BY

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PREFACE.

In compiling the present treatise, I have endeavoured to enunciate clearly the fundamental principles of the theory of Hydromechanics, to explain some of the most important applications of those principles by the help of Mathematical machinery, and to give the student an introduction to some of the recent developments of the science.

It will be seen that I have drawn freely upon the writings of many eminent Mathematicians, who have given attention to various branches of the subject, and I am especially under obligations to Professor Stokes, to Professor Helmholtz, to Sir William Thomson, and to Professor Clerk Maxwell.

Helmholtz and Sir W. Thomson, in particular, have made a romantic advance into a new region of thought, hitherto unexplored, and offering vast and attractive fields of research.

The student will I hope find the following pages a sufficient preparation for the study of the original papers of these Mathematicians, and of others who are engaged in following out the new researches which have been suggested.

I am very much indebted to Mr A. G. Greenhill, Fellow of Emmanuel College and Professor of Mathematics to the advanced class of Artillery Officers at Woolwich, for much kind and valuable assistance. Mr Greenhill has examined most of the proof-sheets, and has given me many important emendations and suggestions throughout the whole work, and particularly in the Chapters on Hydrodynamics, and on Sound.

W. H. BESANT.

St John's College,
January 17, 1877.

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HYDROSTATICS.

CHAPTER I.

1. We learn from common experience that such substances as air and water are characterised by the ease with which portions of their mass can be removed, and by their extreme divisibility. These properties are illustrated by various common facts; if, for instance, we consider the ease with which fluids can be made to permeate each other, the extreme tenuity to which one fluid can be reduced by mixture with a large portion of another fluid, the rarefaction of air which can be effected by means of an air-pump, and other facts of a similar kind, it is clear that, practically, the divisibility of fluid is unlimited: we find, moreover, that in separating portions of fluids from each other, the resistance offered to the division is very slight, and in general almost inappreciable. By a generalization from such observations, the conception naturally arises of a substance possessing in the highest degree these properties, which exist, in a greater or less degree, in every fluid with which we are acquainted, and hence we are led to the following

Definition of a Perfect Fluid.

2. *A perfect fluid is an aggregation of particles which yield at once to the slightest effort made to separate them from each other.*

If then an indefinitely thin plane be made to divide such a fluid in any direction, no resistance will be offered to the division, and the pressure exerted by the fluid on the plane will be entirely normal to it; that is, a perfect fluid is assumed to have no "viscosity," no property of the nature of friction.

The following fundamental property of a fluid is therefore obtained from the above definition.

The pressure of a perfect fluid is always normal to any surface with which it is in contact.

As a matter of fact, all fluids do more or less offer a resistance to separation or division, but, just as the idea of a rigid body is obtained from the observation of bodies in nature which only change form slightly on the application of great force, so is the idea of a perfect fluid obtained from our experiences of substances which possess the characteristics of extremely easy separability and apparently unlimited divisibility.

The following definition will include fluids of all degrees of viscosity.

*A fluid is an aggregation of particles which yield to the slightest effort made to separate them from each other, if it be continued long enough**.

Hence it follows that, in a viscous fluid at rest, there can be no tangential action, or shearing stress, and therefore, as in the case of a perfect fluid,

The pressure of a fluid at rest is always normal to any surface with which it is in contact.

Thus all propositions in Hydrostatics are true for all fluids whatever be the viscosity.

It is in Hydrodynamics that we are limited to the consideration of *perfect* fluids.

3. Fluids are divided into Liquids and Gases; the former, such as water and mercury, are not sensibly compressible, except under very great pressures; the latter are easily compressible, and expand freely if permitted to do so.

Hence the former are sometimes called inelastic, and the latter elastic fluids.

4. Fluids are acted upon by the force of gravity in the same way as solids; with regard to liquids this is obvious; and that air has weight can be shewn directly by weighing a closed vessel, exhausted as far as possible: moreover, the phænomena of the tides shew that fluids are subject to the attractive forces of the sun and moon as well as of the earth, and it is assumed, from these and other similar facts, that fluids of all kinds are subject to the law of gravitation, that is, that they attract, and are attracted by, all other portions of matter, in accordance with that law.

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* See Maxwell's *Heat, Chapters V and XXI.*

Measure of the Pressure of Fluids.

5. Consider a mass of fluid at rest under the action of any forces, and let A be the area of a plane surface exposed to the action of the fluid, that is, in contact with it, and P the force which is required to counterbalance the action of the fluid upon A. If the action of the fluid upon A be uniform, then $\frac{P}{A}$ is the pressure on each unit of the area A. If the pressure be not uniform, it must be considered as varying continuously from point to point of the area A, and if ϖ be the force on a small portion α of the area about a given point, then $\frac{\varpi}{\alpha}$ will approximately express the *rate* of pressure over α. When α is indefinitely diminished let $\frac{\varpi}{\alpha}$ ultimately $= p$, then p is defined to be the measure of the pressure at the point considered, p being the force which would be exerted on an unit of area, if the rate of pressure over the unit were uniform and the same as at the point considered.

The force upon any small area α about a point, the pressure *at* which is p, is therefore $p\alpha + \gamma$, where γ vanishes ultimately in comparison with $p\alpha$ when α (and consequently $p\alpha$) vanishes.

6. In order to employ the principles of Statics in the discussion of the equilibrium of fluids, the following proposition is necessary.

In a mass of fluid at rest any portion may be supposed to become solid without any other change in the circumstances of the equilibrium.

For, if this supposition be made, there will be no alteration in the forces acting on the fluid, and the action between the solidified portion and the rest of the fluid, or between the solidified portion and any surface with which it may be in contact, will be, as before, normal to its surface; the equilibrium of the solid can therefore be considered as maintained by the external forces which act upon it, and the pressure of the remaining fluid.

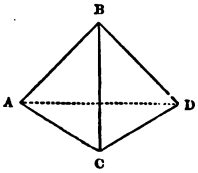
7. *The pressure at any point of a fluid at rest is the same in every direction.*

This is the most important of the characteristic properties of a fluid; it can be deduced from Articles (2) and (6) in the following manner:

Let a small tetrahedron of fluid be supposed solidified; then it is kept at rest by the pressures on its faces, and by the impressed force on its mass.

The former forces depending on the areas of the faces vary as the square, and the latter depending on the volume and density varies as the cube of one of the edges of the solid, which is considered to be homogeneous, and therefore supposing the solid indefinitely diminished, while it retains always a similar form, the latter force vanishes in comparison with the pressures on the faces; and these pressures consequently form a system of forces in equilibrium.

Let p, p' be the rates of pressure on the faces ABC, BCD, and resolve the forces parallel to the edge AD; then, since the projections of the areas ABC, BCD on a plane perpendicular to AD are the same (each equal to α suppose), we have ultimately,



$$p\alpha = p'\alpha,$$
$$\text{or}\quad p = p'.$$

And similarly it may be shewn that the pressures on the other two faces are each equal to p or p'.

As the tetrahedron may be taken with its faces in any direction, it follows that the pressure at a point is the same in every direction.

8. The following proof of the foregoing proposition is taken from Cauchy's *Exercices**.

Let P and Q be two points in a fluid at a finite distance from each other; about PQ as axis describe a cylinder of very small radius, draw a plane through Q perpendicular to QP,

* *Seconde Année*, 1827, p. 23.

draw any plane through P, and suppose the portion of fluid PQ to become solid.

The solid PQ is kept at rest by the pressures on its ends and on its curved surface, and by the impressed forces which act upon it.

Let p, p' be the pressures at Q and P, α the area of the section Q of the cylinder, and α' of the section P; then the pressure $p'\alpha'$ on the end P, resolved parallel to the axis of the cylinder, is equal to $p'\alpha$, and therefore

$$p'\alpha - p\alpha = \text{the impressed force, resolved parallel to } QP.$$

Now whatever be the direction of the plane through P, this impressed force, when the radius of the cylinder is indefinitely diminished, is ultimately equal to the impressed force on the portion QP of the cylinder cut off by a plane through P perpendicular to the axis*, that is, to

$$\int_0^{PQ} f\rho\alpha\, dx,$$

where mf is the force on a particle m of the fluid at a distance x from Q. Hence

$$p' = p + \int_0^{PQ} \rho f dx,$$

or p' is constant for all positions of the plane through P.

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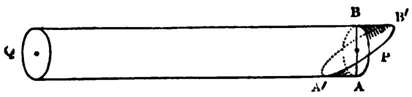
* The following considerations may complete this part of the proof:

Let AB, $A'B'$ be the two planes through P; $\rho\rho'$ the mean densities of APA', BPB'; and ff' the accelerations of the forces which are acting on these portions of fluid.



Then the difference of the forces on QAB and $QA'B'$, (the volumes of which are equal)

$$= \text{the difference of the forces on } APA' \text{ and } BPB'$$
$$= (\rho'f' - \rho f)\,.\,\text{vol}\,.\,APA'$$
$$= \delta(\rho f)\,.\,\frac{2}{3\pi}\alpha AA',$$

and therefore $p' = p + \int_0^{QP} \rho f dx + \frac{2}{3\pi} AA'\,.\,\delta(\rho f)$.

The forces being continuous, the last term is obviously evanescent compared with the other quantities in the equation, and p' is therefore constant.

Transmission of Fluid Pressure.

9. *Any pressure, or additional pressure, applied to the surface, or to any other part, of an incompressible fluid kept at rest, is transmitted equally to all parts of the fluid.*

This property of liquids is a direct result of experiment, and, as such, is sometimes assumed. It is however deducible from our definition of a fluid by aid of the proposition of Art. 6.

Let P be a point in the surface of a liquid at rest, and Q any other point in the liquid; about the straight line PQ describe a cylinder, of very small radius, bounded by the surface at P and by a plane through Q, perpendicular to QP, and suppose this cylinder to become solid.

If the pressure at P be increased by p, the additional force on the cylinder, resolved in the direction of its axis, is $p\alpha$, α being the area of the section of the cylinder perpendicular to its axis, and this must be counteracted by an equal force $p\alpha$ at Q in the direction QP, since the pressure of the liquid on the curved surface is perpendicular to the axis. The pressure at Q is therefore increased by p.

If the straight line PQ do not lie entirely in the liquid, P and Q can be connected by a number of straight lines, all lying in the liquid, and a repetition of the above reasoning will shew that the pressure p is transmitted, unchanged, to the point Q.

10. In consequence of this property, a mass of inelastic fluid can be used as a 'machine' for the purpose of multiplying power.

Thus, if in a closed vessel full of water two apertures be made and pistons A, A' fitted in them, any force P applied to one piston must be counteracted by a force P' on the other piston, such that $P' : P$ in the ratio of the area $A' : A$, for the increased rate of pressure at every point of A is transmitted to every point of A', and the force upon A' depends therefore upon its area*.

* Bramah's Press is an instance of the practical use of this property of liquids.

The action between the two is analogous to the action of a lever, and it is clear that by increasing A' and diminishing A, we can make the ratio $P' : P$ as large as we please.

11. The pressure of an elastic fluid is found to depend upon its density and temperature, as well as upon the nature of the fluid itself.

When the temperature is constant, experiment shews that the pressure varies inversely as the space occupied by the fluids, that is, directly as its density.

This law was first stated by Boyle, but it is a consequence of the more general law that the pressure of a mixture of gases that do not act chemically on each other is the sum of the pressures the gases would exert if they filled the containing vessel separately. For doubling the quantity of gas in the vessel would double the pressure, and a similar proportionate change of pressure would take place for any other change of quantity.

Hence if ρ be the density of a certain quantity of an elastic fluid, and p its pressure, then, as long as the temperature remains the same,

$$p = k\rho,$$

where k is a constant, to be determined experimentally for the fluid at a given temperature.

Measures of Weight, Mass, and Density.

12. The weight, mass, and density of a fluid are measured in the same way as for solid bodies.

If W be the weight of a mass M of fluid, then, in accordance with the usual conventions which define the units of mass and force,

$$W = Mg.$$

If V be the volume of the mass M of fluid of density ρ, then

$$M = \rho V,$$

$$\text{and } \therefore \; W = g\rho V.$$

For the standard substance, $\rho = 1$, and therefore the unit of volume of the standard substance is the unit of mass.

13. In the previous articles no account has been taken of fluids in which the density is variable; but it is easy to conceive the density of a mass of liquid varying continuously from point to point, and it will be hereafter found that a mass of elastic fluid, at rest under the action of gravity, and having a constant temperature throughout, is necessarily heterogeneous: the density at a point of a fluid must therefore be measured in the same way as the pressure at a point, or any other continuously varying quantity.

Measure of the density at any point of a heterogeneous fluid.

Let m be the mass of a volume v of fluid enclosing a given point, and suppose ρ the density of a homogeneous fluid such that the mass of a volume v is equal to m, or such that

$$m = \rho v;$$

then ρ may be defined as the mean density of the portion v of the heterogeneous fluid, and the ultimate value of ρ when v is indefinitely diminished, supposing it always to enclose the point, is the density of the fluid at that point.

EXAMPLES.

(In these Examples g is taken to be 32, when a foot and a second are units.)

1. In a Hydraulic Press the diameter of the ram is nine inches and of the plunger of the pump is one inch; the length of the pump-lever is three feet, and the distance of the point of attachment of the plunger from the fulcrum is nine inches. If a force of 15 lbs. weight be applied at the end of the lever, find the force exerted by the ram of the press.

2. $ABCD$ is a rectangular area subject to fluid pressure; AB is a fixed line, and the pressure on the area is a given function (P) of the length BC (x); prove that the pressure at any point of CD is $\frac{dP}{adx}$, where $a = AB$.

If A be a fixed point, and AB, AD fixed in direction, and if $AB = x$ and $AD = y$, the pressure at $C = \frac{d^2P}{dxdy}$.

3. In the equation $W = g\rho V$, if the unit of force be 100 lbs. weight, the unit of length 2 feet, and the unit of time $\frac{1}{4}$th of a second, find the density of water.

4. If a minute be the unit of time, and a yard the unit of space, and if 15 cubic inches of the standard substance contain 25 oz., determine the unit of force.

5. In the equation, $W = g\rho V$, the number of seconds in the unit of time is equal to the number of feet in the unit of length, the unit of force is 750 lbs. weight, and a cubic foot of the standard substance contains 13500 ounces; find the unit of time.

6. A velocity of four feet per second is the unit of velocity; water is the standard substance and the unit of force is 125 lbs. weight; find the units of time and length.

7. A velocity of 8 feet per second is the unit of velocity, the unit of acceleration is that of a falling body, and the unit of mass is a ton; find the density of water.

8. The density at any point of a liquid, contained in a cone having its axis vertical and vertex downwards, is greater than the density at the surface by a quantity varying as the depth of the point. Shew that the density of the liquid when mixed up so as to be uniform will be that of the liquid originally at the depth of one-fourth of the axis of the cone.

9. The density of a fluid varies from point to point; considering directions proceeding from a given point, prove that the density varies most rapidly along the normal to the surface of equal density containing the point; and of directions in the tangent plane to this surface, the tangents to its principal sections are those in which the rate of variation of density is greatest and least.

CHAPTER II.

THE CONDITIONS OF THE EQUILIBRIUM OF FLUIDS.

14. TAKING the most general case, suppose a mass of fluid, elastic or non-elastic, homogeneous or heterogeneous, to be at rest under the action of given forces, and let it be required to determine the conditions of equilibrium, and the pressure at any point.

Let x, y, z, be the co-ordinates referred to rectangular axes, of any point P in the fluid, and let Q be a point near it, so taken that PQ is parallel to the axis of x.

Take $x + \delta x$, y, z, as the co-ordinates of Q; about PQ describe a small prism or cylinder bounded by planes perpendicular to PQ, and conceive this cylinder to be solidified.

Let α be the area of the section of the cylinder perpendicular to its axis, p the pressure at P, and $p + \delta p$ the pressure at Q.

Then, α being very small the pressure at any point of the plane P will be very nearly equal to p, and the pressure upon it will therefore be

$$(p + \gamma)\,\alpha,$$

where γ vanishes in comparison with p when α is indefinitely diminished.

We can therefore consider α so small that γ may be neglected in comparison with p, and the pressure on the end P of the cylinder may be taken equal to $p\alpha$, and similarly the pressure on the end Q equal to

$$(p + \delta p)\,\alpha.$$

If ρ be the mean density of the cylinder PQ, its mass $= \rho\alpha\delta x$, and $X\rho\alpha\delta x$ will represent the force on PQ parallel to its axis,

if $X\delta m$, $Y\delta m$, $Z\delta m$, be the components of the forces acting on a particle δm of fluid at the point xyz.

Hence, for the equilibrium of PQ,

$$(p + \delta p)\,\alpha - p\alpha = X\rho\alpha\delta x,$$

or $$\delta p = \rho X\delta x.$$

Proceeding to the limit when δx, and therefore δp, is indefinitely diminished, ρ will be the density at P, and we obtain

$$\frac{dp}{dx} = \rho X^*.$$

By a similar process,

$$\frac{dp}{dy} = \rho Y,$$

$$\frac{dp}{dz} = \rho Z.$$

But $$dp = \frac{dp}{dx}dx + \frac{dp}{dy}dy + \frac{dp}{dz}dz;$$

$$\therefore\ dp = \rho\,(Xdx + Ydy + Zdz) \dots\dots\dots\dots\dots (\alpha),$$

the equation which determines the pressure.

15. It is therefore an essential condition of equilibrium that $\rho\,(Xdx + Ydx + Zdz)$ should be a perfect differential of some function $f(x, y, z)$; and

$$\left.\begin{aligned} \therefore\ \frac{d}{dy}(\rho Z) &= \frac{d}{dz}(\rho Y) \\ \frac{d}{dz}(\rho X) &= \frac{d}{dx}(\rho Z) \\ \frac{d}{dx}(\rho Y) &= \frac{d}{dy}(\rho X) \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (\beta),$$

* In the above proof, α is taken so small that its linear dimensions may be neglected in comparison with δx; that is, the change in p, corresponding to a change δx in x, is considered, undisturbed by any alterations in y and z.

from which by differentiating, multiplying the equations respectively by X, Y, and Z, and adding, we obtain

$$X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)=0, \quad \ldots\ldots(\gamma),$$

a necessary condition of equilibrium.

The geometrical interpretation of this equation is that the lines of force,

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

can be intersected orthogonally by a system of surfaces.

16. *Homogeneous Liquids.* If the fluid be homogeneous and incompressible, $Xdx+Ydy+Zdz$ must be a perfect differential (dV) in order that equilibrium may be possible.

In other words, the forces must be such as can be represented by the space-variations of a potential function*.

We then have $$dp=-\rho dV,$$

and $$\therefore \frac{p}{\rho}+V=C.$$

17. If the forces tend to or from fixed centres and are functions of the distances from those centres, we have

$$X=\Sigma\left\{\phi(r)\frac{x-a}{r}\right\},\quad Y=\Sigma\left\{\phi(r)\frac{y-b}{r}\right\},\quad Z=\Sigma\left\{\phi(r)\frac{z-c}{r}\right\},$$

where (a, b, c) are co-ordinates of the centre to which the force $\phi(r)$ tends.

Now $$r^2=(x-a)^2+(y-b)^2+(z-c)^2,$$

$$\therefore Xdx+Ydy+Zdz=\Sigma\phi(r)\,dr,$$

and $$dp=\rho\Sigma\,\phi(r)\,dr.$$

* The meaning here assigned to the word Potential is in accordance with Maxwell's definition, (*Electricity*, Art. 70), which leads to the relations,

$$X=-\frac{dV}{dx},\quad Y=-\frac{dV}{dy},\quad Z=-\frac{dV}{dz}.$$

In this case, since

$$\frac{dX}{dy} = \Sigma \left\{\phi'(r)\frac{x-a}{r}\frac{y-b}{r} - \phi(r)\frac{x-a}{r^2}\frac{y-b}{r}\right\},$$

and $\dfrac{dY}{dx} = \Sigma \left\{\phi'(r)\dfrac{y-b}{r}\dfrac{x-a}{r} - \phi(r)\dfrac{y-b}{r^2}\dfrac{x-a}{r}\right\}$,

it is obvious that the equation (γ) is always satisfied, but it is not to be inferred that the equilibrium of a heterogeneous fluid is always possible with such a system of forces.

When the density is constant, the equations (β) become

$$\frac{dX}{dy} = \frac{dY}{dx}, \quad \frac{dZ}{dy} = \frac{dY}{dz}, \quad \frac{dX}{dz} = \frac{dZ}{dx},$$

which are in this case always satisfied, and therefore the equilibrium of a homogeneous fluid under the action of such forces is always possible.

18. *Elastic Fluids.* When the fluid is elastic, an additional condition is introduced, for, if the temperature be constant,

$$p = k\rho;$$

$$\therefore \frac{dp}{p} = \frac{1}{k}(Xdx + Ydy + Zdz)\dots\dots\dots\dots(\delta).$$

If the forces are derivable from a potential V, *i.e.* if $Xdx + Ydy + Zdz$ be a perfect differential $-dV$,

$$k\frac{dp}{p} = -dV,$$

$$\therefore k \log\frac{p}{C} = -V,$$

$$\text{or } p = C\epsilon^{-\frac{V}{k}}, \text{ and } \rho = \frac{C}{k}\epsilon^{-\frac{V}{k}}.$$

When the forces tend to fixed centres and are functions of the distances, Art. (17), this equation takes the form

$$k\frac{dp}{p} = \Sigma\phi(r)\,dr,$$

and p can be determined.

If the temperature be variable, the relation between the pressure, density, and temperature is found to be

$$p = k\rho(1 + at),$$

where t is the temperature, measured by a Centigrade Thermometer, and $\alpha = \cdot 003665$.

From this we obtain,

$$p = k\rho\alpha\left\{\frac{1}{\alpha} + t\right\} = K\rho T,$$

where $$K = k\alpha, \text{ and } T = \frac{1}{\alpha} + t.$$

T is called the absolute temperature, the zero of which is -273° C.

In this case $\dfrac{dp}{p} = \dfrac{Xdx + Ydy + Zdz}{KT}$,

and, $\therefore$ T must be a function of x, y, z.

19. In any of these cases, if the pressure at any particular point be given, the constant can be determined.

In the case of elastic fluids, if the mass of fluid and the space within which it is contained be given, the constant is determined.

20. The equation for determining p may also be obtained in the following manner.

Let PQ be the axis of a very small cylinder bounded by planes perpendicular to PQ, and imagine this cylinder solidified.

Let p and $p + \delta p$ be the pressures at P and Q, α the areal section, and δs the length of PQ. Then, if $S\delta m$ be the component, in the direction PQ, of the forces acting on an element δm,

$$(p + \delta p)\,\alpha - p\alpha = \rho\alpha S\delta s,$$

and therefore, proceeding to the limit,

$$dp = \rho S ds.$$

That is, the rate of increase of the pressure in any direction is equal to the product of the density and the resolved part of the force in that direction.

If x, y, z be the co-ordinates of P, and X, Y, Z the components of S parallel to the axes,

$$S = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds},$$

and $$\therefore\ dp = \rho\,(Xdx + Ydy + Zdz) \text{ as in Art. 16.}$$

If the position of P be given by the cylindrical co-ordinates r, θ, and z, and if P, T, Z, be the components of S in the directions of r, θ, z,

$$S = P\frac{dr}{ds} + T\frac{rd\theta}{ds} + Z\frac{dz}{ds},$$

and the equation for p becomes

$$dp = \rho\,\{Pdr + Trd\theta + Zdz\}.$$

Again, if the position of P be given by the ordinary polar co-ordinates r, θ, ϕ, and if the components of the force be R, N, and T, in the directions of r, of the perpendicular to the plane of the angle θ, and of the line perpendicular to r in that plane, it will be found that

$$\frac{dp}{\rho} = Rdr + Nr\sin\theta\, d\phi + Trd\theta.$$

In a similar manner the expression for dp may be obtained for any other system of co-ordinates.

21. *Surfaces of equal pressure.* In all cases, in which the equilibrium of the fluid is possible, we obtain by integration

$$p = \phi\,(x, y, z).$$

If p be constant $\qquad \phi\,(x, y, z) = p$ .... (A),

is the equation to the surface at all points of which the pressure is constant, and by giving different values to p we obtain a series of surfaces of equal pressure, and the external surface, or free surface, is obtained by making p equal to the pressure external to the fluid.

If the external pressure be zero the free surface is therefore

$$\phi\,(x, y, z) = 0.$$

The quantities

$$\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz},$$

which are proportional to the direction-cosines of the normal at the point (x, y, z) of the surface A, are equal to

$$\frac{dp}{dx}, \frac{dp}{dy}, \frac{dp}{dz},$$

respectively, i. e. to ρX, ρY, ρZ, and are therefore proportional to X, Y, Z.

Hence the resultant force at any point is in direction of the normal to the surface of equal pressure passing through the point.

The surfaces of equal pressure are therefore the surfaces intersecting orthogonally the lines of force.

It follows from this result that a necessary condition of equilibrium is the existence of a system of surfaces orthogonal to the lines of force, a conclusion derivable also from the equation (γ) of Art. (15), for that equation is the known analytical condition requisite for the existence of such a system.

22. In the particular cases, in which $Xdx + Ydy + Zdz$ is a perfect differential, ρ must be a function of the potential V.

$$\text{For } dp = -\rho dV$$

and dp being a perfect differential, ρ must be a function of V; hence V, and therefore ρ, is a function of p, and surfaces of equal pressure are equipotential surfaces, and are also surfaces of equal density*.

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* These results may also be obtained in the following manner:

Consider two consecutive surfaces of equal pressure, containing between them a stratum of fluid, and let a small circle be described about a point P in one surface, and a portion of the fluid cut out by normals through the circumference. The portion of fluid so cut out may be considered rigid, and kept at rest by the impressed force, and the pressures on its ends and on its circumference. Being very nearly a small cylinder, and the pressures at all points of its circumference being equal, the difference of the pressures on its two faces must be due to the force, which must therefore act in the same direction as these pressures, i. e. in direction of the normal at P.

If the forces are derivable from a potential, the resulting force is perpendicular to the equipotential surfaces, and the surfaces of equal pressure are therefore identical with the equipotential surfaces.

Again, considering the equilibrium of the elemental cylinder, the force acting upon it, per unit of mass, is equal to the difference of potentials divided by the distance between the surfaces of equal pressure, and as the mass of the element is directly proportional to this distance, it follows that the density must be constant, that is, the surfaces of equal pressure are also surfaces of equal density.

If the fluid be elastic and the temperature variable

$$\frac{dp}{p} = -\frac{dV}{KT}.$$

Hence by a similar process of reasoning T is a function of p, and surfaces of equal pressure are also surfaces of equal temperature.

If however $Xdx + Ydy + Zdz$ be not a perfect differential, these surfaces will not in general coincide.

1st. Let the fluid be heterogeneous and incompressible; then the surfaces of equal pressure and of equal density are given respectively by the equations

$$dp = 0, \quad d\rho = 0,$$

or

$$\left.\begin{aligned} Xdx + Ydy + Zdz = 0 \\ \frac{d\rho}{dx}dx + \frac{d\rho}{dy}dy + \frac{d\rho}{dz}dz = 0 \end{aligned}\right\} \text{..............} \quad (B).$$

These then are the differential equations of surfaces which by their intersections determine curves of equal pressure and density.

From (B) we obtain

$$\frac{dx}{Z\frac{d\rho}{dy} - Y\frac{d\rho}{dz}} = \frac{dy}{X\frac{d\rho}{dz} - Z\frac{d\rho}{dx}} = \frac{dz}{Y\frac{d\rho}{dx} - X\frac{d\rho}{dy}} \text{...........} \quad (C).$$

But from the conditions of equilibrium we have

$$\rho\frac{dX}{dy} + X\frac{d\rho}{dy} = \rho\frac{dY}{dx} + Y\frac{d\rho}{dx},$$

$$\rho\frac{dY}{dz} + Y\frac{d\rho}{dz} = \rho\frac{dZ}{dy} + Z\frac{d\rho}{dy},$$

$$\rho\frac{dZ}{dx} + Z\frac{d\rho}{dx} = \rho\frac{dX}{dz} + X\frac{d\rho}{dz},$$

and therefore the equations (C) become

$$\frac{dx}{\frac{dZ}{dy} - \frac{dY}{dz}} = \frac{dy}{\frac{dX}{dz} - \frac{dZ}{dx}} = \frac{dz}{\frac{dY}{dx} - \frac{dX}{dy}} \text{...........} \quad (D),$$

the differential equations of the curves of equal pressure and density.

2nd. Let the fluid be elastic and of variable temperature;

$$\text{then } \frac{dp}{p} = \frac{Xdx + Ydy + Zdz}{KT},$$

and the curves of equal pressure and temperature are given by the simultaneous equations

$$dp = 0, \quad dT = 0;$$

$$\text{or} \quad \left.\begin{aligned} Xdx + Ydy + Zdz = 0 \\ \frac{dT}{dx}dx + \frac{dT}{dy}dy + \frac{dT}{dz}dz = 0 \end{aligned}\right\}.$$

But, since $\frac{dp}{p}$ is a perfect differential, the conditions of equilibrium are in this case

$$\frac{d}{dy}\cdot\frac{Z}{T} = \frac{d}{dz}\cdot\frac{Y}{T}, \text{ \&c.,}$$

or

$$Z\frac{dT}{dy} - Y\frac{dT}{dz} = T\left(\frac{dZ}{dy} - \frac{dY}{dz}\right)$$

$$X\frac{dT}{dz} - Z\frac{dT}{dx} = T\left(\frac{dX}{dz} - \frac{dZ}{dx}\right)$$

$$Y\frac{dT}{dx} - X\frac{dT}{dy} = T\left(\frac{dY}{dz} - \frac{dX}{dy}\right).$$

But, from the preceding equations,

$$\frac{dx}{Z\frac{dT}{dy} - Y\frac{dT}{dz}} = \frac{dy}{X\frac{dT}{dz} - Z\frac{dT}{dx}} = \frac{dz}{Y\frac{dT}{dx} - X\frac{dT}{dy}};$$

$$\therefore \frac{dx}{\frac{dZ}{dy} - \frac{dY}{dz}} = \frac{dy}{\frac{dX}{dz} - \frac{dZ}{dx}} = \frac{dz}{\frac{dY}{dx} - \frac{dX}{dy}},$$

equations of the same form as (D), are in this ease the differential equations of the curves of equal pressure and temperature, and therefore also of equal density.

23. *Fluid at rest under the action of gravity.*

Taking the axis of z vertical, and measuring z downwards,

$$X = 0, \quad Y = 0, \quad Z = g,$$

and the equation (α) becomes

$$dp = g\rho dz,$$

an equation which may also be obtained directly by considering the equilibrium of a small vertical cylinder.

In the case of homogeneous liquid,

$$p = g\rho z + C,$$

and the surfaces of equal pressure are horizontal planes.

Hence the free surface is a horizontal plane, and, taking the origin in the free surface, and Π as the external pressure;

$$\Pi = C,$$

and $$p = g\rho z + \Pi.$$

If there be no pressure on the free surface,

$$p = g\rho z,$$

or *the pressure at any point is proportional to the depth below the surface.*

24. *If two liquids, which do not mix, meet in a bent tube, the heights of the free surfaces above the common surface are inversely as the densities.*

For the pressures at the common surface are the same, and if z, z' be the heights of the upper surfaces above the common surface, and ρ, ρ' the densities, these pressures are respectively

$$g\rho z + \Pi, \quad g\rho' z' + \Pi,$$

$$\text{and } \therefore \frac{z}{z'} = \frac{\rho'}{\rho}.$$

25. In the case of heterogeneous liquid, the equation

$$dp = g\rho dz,$$

shews that ρ must be a function of z. The density and pressure are therefore constant for all points in the same horizontal plane.

As an example, let $\rho \propto z^n = \mu z^n$,

then
$$p = g\mu \frac{z^{n+1}}{n+1} + \Pi.$$

26. It is a well-known law that if a system be in equilibrium under the action of gravity and the pressure of smooth surfaces, the equilibrium is stable, if the centre of gravity be in its lowest possible position.

Hence it follows that, in the case of heterogeneous liquid, the density must increase with the depth, for otherwise the equilibrium would be unstable.

Thus, if heterogeneous liquid be poured from one vessel to another, it will settle with the heaviest strata lowest, the law of density of course being changed.

A quantity of liquid, the density of which is a given function of the depth, is contained in a vessel of given shape; if the liquid be transferred to another vessel, it is required to find the new law of density, each vessel being in the form of a surface of revolution with its axis vertical.

Measuring x upwards from the lowest point of the liquid, let $y = f(x)$ be the generating curve of the first vessel, and $y = \phi(x)$ of the second.

Then, if the stratum at the height x in the first vessel correspond to the stratum at the height x' in the second, we obtain, since the volumes are equal,

$$\int_0^x \{f(\xi)\}^2 d\xi = \int_0^{x'} \{\phi(\xi)\}^2 d\xi,$$

and performing the integrations, we find x in terms of x', and therefore ρ, which is a given function of x, becomes a new function of x'.

Moreover, if h and h' be the depths of the liquid in the two vessels, h is given in terms of h', and therefore the density, ρ, can be found in terms of $h' - x'$, the depth.

If the new law of density be given, and it be required to find the shape of the new vessel, we may proceed as follows:

The density being a given function of $h - x$, and also of $h' - x'$, we can, by equating the two expressions, find x in terms of x'.

Also, differentiating with regard to x' the above equation, we obtain

$$y^2 \frac{dx}{dx'} = y'^2,$$

which at once, by substituting for x its value in terms of x', gives the equation required. The value of h' will be then obtained by equating to each other the whole volumes.

EXAMPLE. *The density of a liquid in a cylindrical vessel varies as the depth; find the new law of density if the liquid be poured into a conical vessel having its vertex downwards.*

In this case $\rho = \mu(h - x)$,

and $\pi a^2 x = \frac{1}{3}\pi x'^3 \tan^2 \alpha$;

also $\pi a^2 h = \frac{1}{3}\pi h'^3 \tan^2 \alpha$;

$$\therefore \rho = \mu \tan^2 \alpha \frac{h'^3 - x'^3}{3a^2} = \frac{\mu \tan^2 \alpha}{3a^2}(3h'^2 z - 3h'z^2 + z^3),$$

if z be the depth.

27. *Elastic fluid at rest under the action of gravity.*

In this case, $p = k\rho$,

and $\frac{dp}{p} = \frac{g}{k} dz$;

$$\therefore \log \frac{p}{C} = \frac{gz}{k} \text{ and } p = C\epsilon^{\frac{gz}{k}}.$$

The surfaces of equal pressure are in this case also horizontal planes, and the constant C must be determined by a knowledge of the pressure for a given value of z, or by some other fact in connection with the particular case.

EXAMPLE. *A closed cylinder, the axis of which is vertical, contains a given mass of air.*

Measuring z from the top of the cylinder,

$$\rho = \frac{p}{k} = \frac{C}{k} \epsilon^{\frac{gz}{k}},$$

$\therefore$ if M be the given mass, a the radius, and h the height of the cylinder,

$$M = \int_0^h \rho \pi a^2 dz = \pi a^2 \frac{C}{g} (\epsilon^{\frac{gh}{k}} - 1),$$

whence C is determined.

28. *Illustrations of the use of the general equation.*

(1) Let a given volume V of liquid be acted upon by forces

$$-\frac{\mu x}{a^2}, \quad -\frac{\mu y}{b^2}, \quad -\frac{\mu z}{c^2},$$

respectively parallel to the axes;

then $$dp = \rho \left(-\frac{\mu x}{a^2} dx - \frac{\mu y}{b^2} dy - \frac{\mu z}{c^2} dz\right),$$

and $$p = C - \frac{\mu \rho}{2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right).$$

The surfaces of equal pressure are therefore similar ellipsoids, and the equation to the free surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2C}{\mu \rho},$$

assuming that there is no external pressure.

The condition which determines the constant is that the volume of the fluid is given, and we have

$$V = \frac{4}{3} \pi abc \,.\left(\frac{2C}{\mu \rho}\right)^{\frac{3}{2}},$$

$$\text{and } C = \frac{\mu \rho}{2} \,.\left(\frac{3V}{4\pi abc}\right)^{\frac{2}{3}}.$$

(2) *A given volume of heavy liquid is at rest under the action of a force to a fixed point varying as the distance from that point.*

Take the fixed point as origin, and measure z vertically downwards;

then $$X = -\mu x,\ Y = -\mu y, \text{ and } Z = g - \mu z;$$

$$\therefore\ dp = \rho\{-\mu x dx - \mu y dy + (g - \mu z)dz\},$$

and $$\frac{p}{\rho} = C - \mu\frac{x^2 + y^2 + z^2}{2} + gz.$$

The surfaces of equal pressure are spheres, and the free surface supposing the external pressure zero, is given by the equation

$$x^2 + y^2 + z^2 - \frac{2g}{\mu}z = \frac{2C}{\mu}.$$

The volume of this sphere is

$$\frac{4}{3}\pi\left(\frac{2C}{\mu} + \frac{g^2}{\mu^2}\right)^{\frac{3}{2}};$$

equating this to the given volume, the constant C is determined, and the pressure at any point is then given in terms of r and z.

Rotating Fluid.

29. If a quantity of fluid revolve uniformly and without any relative displacement of its particles, about a fixed axis, the preceding equations will enable us to determine the pressure at any point, and the nature of the surfaces of equal pressure.

For, in such cases of relative equilibrium, every particle of the fluid moves uniformly in a circle, and the resultant of the external forces acting on any particle m of the fluid, and of the fluid pressure upon it, must be equal to a force $m\omega^2 r$ *towards* the axis, ω being the angular velocity, and r the distance of m from the axis; it follows therefore that the external forces, combined with the fluid pressures and forces $m\omega^2 r$ acting *from* the axis, form a system in statical equilibrium, to which the equations of the previous articles are applicable.

A mass of homogeneous liquid, contained in a vessel, revolves uniformly about a vertical axis; required to determine the pressure at any point, and the surfaces of equal pressure.

Take the vertical axis as the axis of z; then, resolving the force $m\omega^2 r$ parallel to the axes, its components are $m\omega^2 x$ and $m\omega^2 y$, and the general equation of fluid equilibrium becomes

$$dp = \rho\,(\omega^2 x dx + \omega^2 y dy - g dz),$$

and therefore

$$p = \rho\{\tfrac{1}{2}\omega^2(x^2 + y^2) - gz\} + C.$$

The surfaces of equal pressure are therefore paraboloids of revolution, and if the vessel be open at the top, the free surface is given by the equation

$$\omega^2(x^2 + y^2) - 2gz + \frac{2C}{\rho} = \frac{2\Pi}{\rho},$$

where Π is the external pressure.

The constant must be determined by help of the data of each particular case.

For instance, let the vessel be closed at the top and be just filled with liquid, and let $\Pi = 0$; then, taking the origin at the highest point of the axis, $p = 0$ when x y and z vanish, and therefore $C = 0$, and

$$p = \rho\,\{\tfrac{1}{2}\omega^2(x^2 + y^2) - gz\}.$$

30. Next consider the case of elastic fluid enclosed in a vessel which rotates about a vertical axis;

as before $$dp = \rho\,\{\omega^2(x dx + y dy) - g dz\},$$

$$\text{and } p = k\rho;$$

$$\therefore\ k \log \rho = \omega^2 \frac{x^2 + y^2}{2} - gz + C,$$

so that the surfaces of equal pressure and density are paraboloids.

Let the containing vessel be a cylinder rotating about its axis, and suppose the whole mass of fluid given; then, to determine the constant, consider the fluid arranged in elementary horizontal rings each of uniform density: let r be the radius of one of these rings at a height z, δr its horizontal and δz

its vertical thickness, h the height, and a the radius of the cylinder:

$$\text{the mass of the ring} = 2\pi\rho r\delta r\delta z,$$

$$\text{and the whole mass } (M) \text{ of the fluid} = \int_0^h \int_0^a 2\pi\rho r dr dz,$$

the origin being taken at the base of the cylinder.

$$\text{Now } \rho = \epsilon^{\frac{C}{k}} . \epsilon^{\frac{\omega^2 r^2 - 2gz}{2k}};$$

$$\text{and } \therefore\ M = \frac{2\pi k^2}{g\omega^2} \epsilon^{\frac{C}{k}} (\epsilon^{\frac{\omega^2 a^2}{2k}} - 1)(1 - \epsilon^{-\frac{gh}{k}}),$$

an equation by which C is determined.

31. In general the equation of equilibrium for a fluid revolving uniformly and acted upon by forces of any kind, is

$$dp = \rho \{Xdx + Ydy + Zdz + \omega^2 (xdx + ydy)\}.$$

In order that the equilibrium may be possible, three equations of condition must be satisfied, expressing that dp is a perfect differential, and, if these conditions are satisfied, the surfaces of equal pressure, and, in certain cases, the free surface can be determined; but it must be observed that a free surface is not always possible. In fact, in order that there may be a free surface, the surfaces of equal pressure must be symmetrical with respect to the axis of rotation.

EXAMPLE. *A closed vessel is completely filled with homogeneous liquid, which is made to rotate uniformly about an axis inclined to the vertical, required to find the surfaces of equal pressure.*

Let α be the inclination of the axis to the vertical, and take the axis of x in the vertical plane through the axis of rotation; then

$$\frac{1}{\rho} dp = (\omega^2 x - g \sin \alpha) dx + \omega^2 y dy - g \cos \alpha \, dz,$$

$$\frac{p}{\rho} = \tfrac{1}{2}\omega^2 (x^2 + y^2) - gx \sin \alpha - gz \cos \alpha + C,$$

and the required surfaces are paraboloids having their common axis parallel to the axis of revolution.

It will be seen that in this case the pressure *about* any given particle of fluid varies with its position in the circle in which it is moving; in other words, a given particle of fluid passes across different surfaces of equal pressure in the course of its revolution.

Whole Pressure.

32. DEF. *The whole pressure of a fluid on any surface with which it is in contact is the sum of the normal pressures on each of its elements.*

If then p be the pressure at a point of an element δS of the surface,

$$p\delta S \text{ is the pressure on the element,}$$

and $\iint p dS$ is the whole pressure, the summation extending over the whole of the surface considered.

If the fluid be homogeneous liquid, and gravity the only force in action, $p = g\rho z$, measuring z vertically downwards from the surface of the liquid,

$$\text{and } \iint p dS = \iint g\rho z dS.$$

Let $\bar{z}$ be the depth of the centre of gravity of the *surface* S,

$$\text{then } \bar{z} \,.\, S = \iint z dS;$$

$$\text{and } \therefore \text{ the whole pressure} = g\rho\bar{z}S,$$

i.e. the whole pressure is equal to the weight of a cylindrical column of fluid, the height of which is $\bar{z}$, and the base a plane area equal to the area of the surface.

We now add some examples of the determination of whole pressure.

(1) *A hemispherical bowl filled with water.*

Let r be its radius, ρ the density of water.

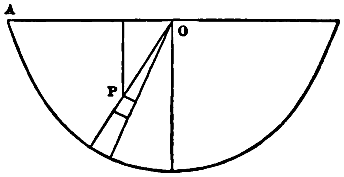
Then the surface $= 2\pi r^2$,

$$\text{and } \bar{z} = \frac{r}{2};$$

$$\therefore \text{ whole pressure} = g\rho\pi r^3,$$

i.e. whole pressure : the weight of the fluid :: 3 : 2.

(2) *The density of a heavy liquid varies as the square of the depth; it is required to find the whole pressure on a semicircular area immersed vertically with its bounding diameter in the surface.*



Let $OP = r$, $AOP = \theta$; then (Art. 25) if the density $= \mu\,(\text{depth})^2$, the pressure at P

$$= \frac{\mu g}{3}(r \sin\theta)^3,$$

and the whole pressure

$$= 2\int_0^a \int_0^{\frac{\pi}{2}} \frac{\mu g}{3} r^4 \sin^3\theta d\theta dr,$$

$$= \frac{4}{45}\mu g a^5.$$

(3) *A cylindrical vessel is closed at the top, and very nearly filled with incompressible fluid, which rotates uniformly about the axis of the cylinder; to find the whole pressure on the curved surface and on the top of the cylinder.*

In this case, taking the centre of the top as origin, and measuring z downwards,

$$\frac{p}{\rho} = \frac{\omega^2 r^2}{2} + gz.$$

Let a be the radius of the cylinder, h its height; then at a depth z, the pressure at its surface

$$= \rho\left(\frac{\omega^2 a^2}{2} + gz\right),$$

an element of surface $= 2\pi a \,.\, \delta z$;

$$\therefore \text{ the whole pressure on the curved surface}$$

$$= \int_0^h 2\pi a\rho\left(\tfrac{1}{2}\omega^2 a^2 + gz\right) dz,$$

$$= \pi\rho a^3 h\omega^2 + \pi\rho a g h^2.$$

The pressure on the top at a distance r from the origin $= \frac{1}{2}\rho\omega^2 r^2$,

and an element of its area $= 2\pi r\delta r$;

therefore the whole pressure on the top

$$= \int_0^a \pi\rho\omega^2 r^3 dr = \tfrac{1}{4}\pi\rho\omega^2 a^4.$$

(4) *A hollow spherical shell is just filled with homogeneous liquid, and the liquid is at rest under the action of a force, to a point on the inner surface of the shell, proportional to the distance from that point; it is required to find the whole pressure on the shell.*

Let O be the centre of force, and r the distance of any point from O.

Then $$dp = -\mu\rho r dr,$$

and $$p = C - \mu\rho\frac{r^2}{2}.$$

The pressure vanishes at the other extremity of the diameter OA, and therefore

$$p = \mu\rho\left(2a^2 - \frac{r^2}{2}\right),$$

a being the radius AC.

If P be a point in the sphere and $ACP = \theta$,

then $OP = 2a\cos\dfrac{\theta}{2}$,

and the pressure at $P = 2\mu\rho a^2 \sin^2\dfrac{\theta}{2}$.

If $PCQ = \delta\theta$, in the plane of θ, the surface generated by the revolution of the arc PQ about OA

$$= 2\pi a\delta\theta \,.\, a\sin\theta,$$

and $\therefore$ the whole pressure on the surface

$$= \int_0^\pi 4\pi\mu\rho a^4 \sin^2\frac{\theta}{2}\sin\theta d\theta$$

$$= 2\pi\mu\rho a^4 \int_0^\pi (1 - \cos\theta)\sin\theta d\theta$$

$$= 4\pi\mu\rho a^4.$$

EXAMPLES.

1. A rectangular area $ABCD$ is just immersed in water with the side AB in the surface; find a point P in AB such that the pressure on the triangle APD may be one-fourth of the pressure on the rectangle.

2. The side AB of a triangle ABC is in the surface of a fluid, and points D, E, are taken in AC, such that the pressures on the triangles BAD, BDE, BEC, are equal; find the ratios

$$AD : DE : EC.$$

3. A triangle ABC is immersed in fluid, in such a position that the point A is in the surface and the lines AB, AC, are equally inclined to it; BC being produced to meet the surface in E, shew that the pressures on the triangles ABC, ACE, are in the ratio

$$AB^2 - AC^2 : AC^2.$$

4. The density of a liquid varies as the square of the depth below the surface; find the whole pressures, 1st, on a rectangular area just immersed vertically with one side in the surface, 2nd, on a circular area just immersed.

5. A parabolic area, bounded by the latus rectum, is just immersed vertically, with its vertex in the surface of a liquid; find the whole pressure upon it, 1st, when the liquid is homogeneous, 2nd, when its density varies as the depth.

6. A solid cone is completely immersed in water with a generating line vertical, and its vertex in the surface; compare the whole pressures on the curved surface, and the base.

7. Find the surfaces of equal pressure when the forces tend to fixed centres and vary as the distances from those centres.

8. A regular tetrahedron is filled with fluid, and held so that two of its opposite edges are horizontal; compare the pressures on its several sides with the weight of the fluid.

9. A spherical mass of elastic fluid is compressed into the cube which can be inscribed within the sphere; compare the whole pressures on the surfaces of the cube and sphere.

If a mass of air in a cubical vessel be compressed into the sphere which can be inscribed in the cube, the whole pressures on the two surfaces are equal.

10. A thin tube in the form of an isosceles triangle is just filled with three liquids which do not mix; and, when held with its base vertical, the three points of junction bisect the sides; prove that the densities are in arithmetic progression.

11. In a solid sphere two spherical cavities, whose radii are equal to half the radius of the solid sphere, are filled with liquid; the solid and liquid particles attract each other with forces which vary as the distance: prove that the surfaces of equal pressure are spheres concentric with the solid sphere.

12. A given quantity of elastic fluid is contained in a hollow sphere, and its particles are acted upon by a force to the centre of the sphere varying inversely as the distance. The sphere being supposed to vary in size, shew that the whole pressure on its surface varies inversely as its radius, provided $\mu < 3\kappa$, where μ is the absolute force, and κ the ratio of the pressure to the density of the fluid.

13. A quantity of incompressible fluid within a cylinder is acted upon by a force to a point in its axis varying directly as the distance, and is made to rotate uniformly about the axis. Taking no account of gravity, determine the nature of the free surfaces for different angular velocities; and in particular, find the angular velocity for which the free surface will be that of a cone.

14. A closed cylindrical vessel is very nearly filled with incompressible fluid, which is acted upon by a force, varying as the distance, to the middle point of the axis of the cylinder; if $2a$ be the length of the axis and c the radius of either end, shew that the whole pressure on the curved surface : the whole pressure on the ends $:: 8a^2 : 3c^2$.

Also find this ratio when the centre of force is at the centre of either end of the cylinder.

15. A mass of fluid rests upon a plane subject to a central attractive force $\left(\frac{\mu}{r^2}\right)$, situated at a distance c from the plane on the side opposite to that on which the fluid is; and a is the radius of the free

spherical surface of the fluid: shew that the whole pressure on the plane

$$=\frac{\pi\rho\mu}{a}(a-c)^2.$$

16. Find the surfaces of equal pressure for fluid acted upon by two forces which vary as the inverse square of the distance from two fixed points.

Prove that if the surface of no pressure be a sphere, the loci of points at which the pressure varies inversely as the distance from one of the centres of force are also spheres.

17. The unit of velocity being a velocity of one foot per second, and the unit of acceleration that of a falling body, find the gravitation unit of force in the equation $p=g\rho z$, water being taken as the standard substance.

18. A cylindrical rod, of radius one inch and length eight inches, is placed in a vessel of water ten inches deep, with one end on the bottom of the vessel, and is inclined to the vertical at an angle 45°; if an inch be the unit of length, a yard per second the unit of velocity, and the density of water the unit of density, find the number of gravitation units of force in the whole pressure on the rod.

19. A vertical cylinder contains water which is made to rotate uniformly about the axis; if $\frac{1}{n}$th of the axis be above the surface when there is no rotation, prove that the greatest angular velocity which can be imparted to the liquid without causing any of it to leave the cylinder, is $\frac{2}{a}\sqrt{\frac{gh}{n}}$, h being the height and a the radius of the cylinder.

20. A conical vessel, of which the vertical angle is 60°, is placed with its axis vertical and vertex downwards, and half filled with water; prove that the greatest angular velocity about the axis which the water can have without overflowing is $\sqrt{\frac{2g}{h}}$, where h is the height of the cone.

21. A closed cylinder, with its axis vertical, is just filled with liquid which rotates uniformly about a generating line; find the whole pressures on the base, the upper end, and the curved surface.

22. A vessel in the form of an inverted cone is partly filled with fluid, and closed with a lid; it is then made to revolve uniformly about its axis; if a small hole be now made at the vertex, determine how much of the fluid will escape, considering the different cases that arise according to the magnitude of the angular velocity. If this be indefinitely increased, prove that the surface of the fluid is a circular cylinder, and find its radius.

23. If the force at any point is given by a potential ϕ, and if a tube of small but variable circular section be imagined in the liquid, the whole pressure upon which is P, prove that

$$\frac{d^2P}{d\phi ds} + 2\pi\rho r = 0$$

where r is the radius of the section, and s is measured along the axis of the tube.

24. The density of a liquid, contained in a cylindrical vessel, varies as the depth; it is transferred to another vessel, in which the density varies as the square of the depth; find the shape of the new vessel.

25. A quantity of liquid, the density of which varies as the depth, fills an inverted paraboloid, of latus rectum c, to a height h; find the shape of a vessel, in the form of a surface of revolution, such that if this liquid be poured into it its density will vary as the square of its depth.

26. A circular cone, of vertical angle $\frac{\pi}{3}$, is just filled with water, and has a generating line rigidly attached to a horizontal plane. The plane is caused to revolve with uniform angular velocity about a vertical axis through the apex of the cone: find the greatest velocity which will allow of the pressure being zero at the highest point; and in this case find the whole pressure on the base.

27. A straight rod, every particle of which attracts with a force varying inversely as the square of the distance, is surrounded by a mass of homogeneous incompressible fluid; find the form of the surfaces of equal pressure.

28. Water in a vessel completely full is made to rotate uniformly about a horizontal axis; find the surfaces of equal pressure.

29. A quantity of heavy liquid is attracted to a fixed centre, by a constant force the intensity of which is equal to the force of

gravity, and is supported by a horizontal plane. Find the form of the surfaces of equal pressure; and also the pressure on the plane, proving that when the plane passes through the centre of force it is equal to four-thirds of the weight of the liquid. Find also expressions for the pressure on the plane when it is either above or below the centre of force.

30. A rigid spherical shell is filled with homogeneous inelastic fluid, every particle of which attracts every other with a force varying inversely as the square of the distance; shew that the difference between the pressures at the surface and at any point within the fluid varies as the area of the least section of the sphere through the point.

31. At the vertex of a solid cone (vertical angle $2a$) there is a centre of force the attraction to which varies as the distance; and a given quantity of liquid is in equilibrium under the action of this force alone. Determine the form of its free surface. If the volume of the liquid be $\frac{4}{3}\pi a^3 \cos^2 \frac{a}{2}$, prove that the whole pressure on the surface of the cone $= \frac{1}{4}\mu\rho\pi a^4 \sin a$, where ρ is the density of the liquid and μ the absolute force.

32. An open vessel containing liquid is made to revolve about a vertical axis with uniform angular velocity. Find the form of the vessel and its dimensions in order that it may be just emptied.

33. A quantity of liquid (gravity being supposed not to act) just fills a hollow sphere, and is repelled from a point in the surface of the sphere by a force $= \mu \times$ distance: if the liquid revolve round the diameter passing through the centre of force with uniform angular velocity ω, find the whole pressure on the surface of the sphere. If, by diminishing the angular velocity one half, the pressure is also diminished one half, shew that $\omega^2 = 6\mu$.

34. A rectangular plate of thin metal of given size is bent and held so that two opposite edges are parallel and in the same horizontal plane, and the vertical ends are then closed by flat plates; if this vessel be filled with water, find its form when the whole pressure upon its curved surface is a maximum.

35. An infinite mass of homogeneous fluid surrounds a closed surface and is attracted to a point (O) within the surface with a force which varies inversely as the cube of the distance. If the pressure on any element of the surface about a point P be resolved along PO,

prove that the whole radial pressure, thus estimated, is constant, whatever be the shape and size of the surface, it being given that the pressure of the fluid vanishes at an infinite distance from the point O.

36. A vessel formed by the revolution of a cardioid $r=a(1-\cos\theta)$ about its axis which is vertical (vertex upwards) is just filled with water and rotates about that axis with uniform angular velocity. Find this velocity, when the line of no pressure is given by $\theta=\frac{\pi}{6}$. Find also the pressure at any other point, and the points of maximum pressure.

37. A closed vessel full of liquid is made to revolve with uniform angular velocity ω about a vertical axis through its highest point; shew that the total pressure of the liquid on the surface is increased by $\frac{1}{2}Ak^2\rho\omega^2$; A being the area of the surface, and k the radius of gyration of the surface about the vertical axis.

38. All space being supposed filled with an elastic fluid the particles of which are attracted to a given point by a force varying as the distance, and the whole mass of the fluid being given, find the pressure on a circular disc placed with its centre at the centre of force.

39. A thin ellipsoidal shell, attracting according to the law of nature, is surrounded by homogeneous liquid; find the surfaces of equal pressure, neglecting the attraction of the liquid on itself.

40. Circles are drawn having their centres on the axis of z and touching at the origin the plane xy, and the position of a point P is defined by r, θ, ϕ, where r is the radius of the circle through P, centre C, θ is the angle OCP, and ϕ the inclination of the plane OCP to a fixed plane through the axis of z; prove that

$$\frac{dp}{\rho}=R(1-\cos\theta)\,dr+T\sin\theta\,dr+Trd\theta+Nr\sin\theta\,d\phi,$$

where mR, mT, mN are the forces, on an element m of liquid at P, along CP, along the tangent to the circle at P, and perpendicular to the plane of the circle.

41. A mass m of elastic fluid is rotating about an axis with uniform angular velocity ω, and is acted on by an attraction towards a point in that axis equal to μ times the distance, μ being greater than ω^2; prove that the equation of a surface of equal density ρ is

$$\mu(x^2+y^2+z^2)-\omega^2(x^2+y^2)=k\log\left\{\frac{\mu(\mu-\omega^2)^2}{8\pi^3}\cdot\frac{m^2}{\rho^2k^3}\right\}.$$

CHAPTER III.

THE RESULTANT PRESSURE OF FLUIDS ON SURFACES.

33. IN the preceding Chapter we have shewn how to investigate the pressure *at any point* of a fluid at rest under the action of given forces; we now proceed to determine the resultants of the pressures exerted by fluids *upon surfaces* with which they are in contact.

We shall consider, first, the action of fluids on plane surfaces, secondly, of fluids under the action of gravity upon curved surfaces, and thirdly, of fluids at rest under any given forces upon curved surfaces.

Fluid Pressures on Plane Surfaces.

The pressures at all points of a plane being perpendicular to it, and in the same direction, the resultant pressure is equal to the sum of these pressures, that is, to the whole pressure, and acts in the same direction.

Hence, if the fluid be incompressible and acted upon by gravity only, the resultant pressure on a plane

$$= \text{the whole pressure}$$
$$= g\rho\bar{z}A,$$

where A is the area and $\bar{z}$ the depth of the centre of gravity.

In general, if the fluid be of any kind, and at rest under the action of any given forces, take the axes of x and y in the plane, and let p be the pressure at the point (x, y).

The pressure on an element of area $\delta x \delta y = p\delta x \delta y$;

$$\therefore \text{ the resultant pressure} = \iint p\,dy\,dx,$$

the integration extending over the whole of the area considered.

If polar co-ordinates be used, the resultant pressure is given by the expression

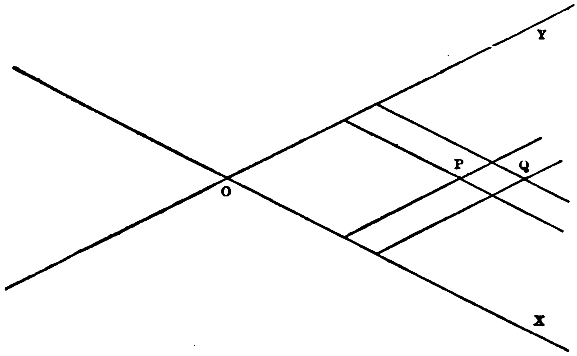
$$\iint pr\,dr\,d\theta.$$

34. DEF. *The centre of pressure is the point at which the direction of the single force, which is equivalent to the fluid pressures on the plane surface, meets the surface.*

The *centre of pressure* is here defined with respect to plane surfaces only; it will be seen afterwards that the resultant action of fluid on a curved surface is not always reducible to a single force.

In the case of a *heavy* fluid, it is clear that the centre of pressure of a horizontal area, the pressure on every point of which is the same, is its centre of gravity; and, since pressure increases with the depth, the centre of pressure of any plane area, not horizontal, is below its centre of gravity.

PROP. *To obtain formulæ for the determination of the centre of pressure of any plane area.*



Let p be the pressure at the point (x, y), referred to rectangular axes in the plane, $x + \delta x$, $y + \delta y$, the co-ordinates of Q, $\bar{x}$, $\bar{y}$, co-ordinates of the centre of pressure.

Then $\bar{y}\,.\iint p\,dy\,dx$ = moment of the resultant pressure about OX,

= the sum of the moments of the pressures on all the elements of area about OX,

$$= \Sigma\, p\delta y\,\delta x\,.\,y$$

$$= \iint py\,dy\,dx;$$

$$\therefore\ \bar{y} = \frac{\iint py\,dy\,dx}{\iint p\,dy\,dx},$$

and similarly $\bar{x} = \dfrac{\iint px\,dy\,dx}{\iint p\,dy\,dx}$,

the integrals being taken so as to include the area considered.

If polar co-ordinates be employed, a similar process will give the equations

$$\bar{x} = \frac{\iint pr^2\cos\theta\,dr\,d\theta}{\iint pr\,dr\,d\theta},\quad \bar{y} = \frac{\iint pr^2\sin\theta\,dr\,d\theta}{\iint pr\,dr\,\delta\theta}.$$

35. If the fluid be homogeneous and inelastic, and if gravity be the only force in action,

$$p = g\rho h,$$

where h is the depth of the point P below the surface; and we obtain

$$\bar{x} = \frac{\iint hx\,dy\,dx}{\iint h\,dy\,dx},\quad \bar{y} = \frac{\iint hy\,dy\,dx}{\iint h\,dy\,dx}\ \ldots\ldots\ldots\ldots\ldots (\alpha).$$

It is sometimes useful to take for one of the axes the line of intersection of the plane with the surface of the fluid: if we take this line for the axis of x, and θ as the inclination of the plane to the horizon, $p = g\rho y\sin\theta$, and therefore

$$\bar{x} = \frac{\iint xy\,dy\,dx}{\iint y\,dy\,dx},\quad \bar{y} = \frac{\iint y^2\,dy\,dx}{\iint y\,dy\,dx}\ \ldots\ldots\ldots\ldots\ldots (\beta).$$

From these last equations (β) it appears that the position of the centre of pressure is independent of the inclination of the plane to the horizon, so that if a plane area be immersed in fluid, and then turned about its line of intersection with the surface as a fixed axis, the centre of pressure will remain unchanged.

If in the equations (α) we make h constant, that is, if we suppose the plane horizontal, $\bar{x}$ and $\bar{y}$ are the co-ordinates of the centre of gravity of the area, a result in accordance with Art. (36); but, in the equations (β), the values of $\bar{x}$ and $\bar{y}$ are independent of θ, and are therefore unaffected by the evanescence of θ. This apparent anomaly is explained by considering that, however small θ be taken, the portion of fluid between the plane area and the surface of the fluid is always wedge-like in form, and the pressures at the different points of the plane, although they all vanish in the limit, do not vanish in ratios of equality, but in the constant ratios which they bear to one another for any finite value of θ*.

36. The following theorem determines geometrically the position of the centre of pressure for the case of a heavy liquid.

If a straight line be taken in the plane of the area, parallel to the surface of the liquid and as far below the centre of inertia of the area as the surface of the liquid is above; the pole of this straight line with respect to the momental ellipse at the centre of inertia whose semi-axes are equal to the principal radii of gyration at that point will be the centre of pressure of the area.

Taking A for the area, and b, a for the principal radii of gyration, these are determined by the equations

$$Ab^2 = \iint y^2\, dxdy, \quad Aa^2 = \iint x^2\, dxdy,$$

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* The equations of this article may be obtained by the following reasoning, which, as a slightly different method, it may be perhaps useful to insert.

Through the boundary line of the plane area draw vertical lines to the surface, and let the fluid so enclosed be considered solid; then the reaction of the plane, resolved vertically, is equal to the weight of the solidified fluid, which acts in a vertical line through its centre of gravity; and the point in which this line meets the plane is the centre of pressure.

Taking the same axes as in (38), the weight of an elementary prism, acting through the point x, y, is $g\rho h\delta x\delta y \cos\theta$, where θ is the inclination of the plane to the horizon; and therefore the *centre* of these parallel forces (Todhunter's *Statics*, Art. 66) acting at points of the plane, is given by the equations

$$\bar{x} = \frac{\iint g\rho h\, x\cos\theta\, dy\, dx}{\iint g\rho h\cos\theta\, dy\, dx}, \quad \bar{y} = \frac{\iint g\rho h\, y\cos\theta\, dy\, dx}{\iint g\rho h\cos\theta\, dy\, dx},$$

or

$$\bar{x} = \frac{\iint hx\, dy\, dx}{\iint h\, dy\, dx}, \quad \bar{y} = \frac{\iint hy\, dy\, dx}{\iint h\, dy\, dx}.$$

and the equation of the momental ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let $\bar{x}$, $\bar{y}$ be co-ordinates of the centre of pressure, and

$$x\cos\alpha+y\sin\alpha=p$$

the equation to the line in the surface;

then $$\bar{x}=\frac{\iint(p-x\cos\alpha-y\sin\alpha)\,xdxdy}{\iint(p-x\cos\alpha-y\sin\alpha)\,dxdy}=-\frac{a^2}{p}\cos\alpha,$$

and similarly, $$\bar{y}=-\frac{b^2}{p}\sin\alpha;$$

$\therefore$ $(\bar{x}, \bar{y})$ is the pole of the line

$$x\cos\alpha+y\sin\alpha=-p$$

with respect to the momental ellipse.

37. Ex. 1. *A given volume of liquid is at rest on a fixed plane, under the action of a force, to a fixed point in the plane, varying as the distance; required to find the pressure on the plane.*

Taking the fixed point as origin, the expression for the pressure at any point is

$$p=C-\tfrac{1}{2}\mu\rho\,(x^2+y^2+z^2)=C-\tfrac{1}{2}\mu\rho r^2,$$

where r is the distance from the origin; and if $\frac{2}{3}\pi a^3$ be the given volume, the free surface is a hemisphere of radius a, and

$$p=\tfrac{1}{2}\mu\rho\,(a^2-r^2).$$

The portion of the plane in contact with fluid is a circle of radius a, and therefore the pressure upon it

$$=\int_0^{2\pi}\int_0^a pr\,dr\,d\theta$$

$$=\tfrac{1}{4}\pi\mu\rho a^4.$$

This result may be written in the form $\mu\frac{3}{8}a\,.\,\frac{2}{3}\pi\rho a^3$, which is the expression for the attraction on the whole mass of fluid, supposed to be condensed into a material particle at its centre of gravity, and might in fact have been at once obtained by considering the fluid solidified, and kept at rest by the attraction to the centre of force and the reaction of the plane. (See Todhunter's *Statics*, Art. 220.)

Ex. 2. *A rectangle has two sides horizontal, to find its centre of pressure.*

Take the upper side for axis of y and its middle point as origin; let a, b, be the sides of the rectangle, c the depth of the origin, and θ the inclination to the horizon of the plane of the rectangle.

Divide the rectangle into horizontal strips, and let x be the distance of one of these from the origin; then its depth is

$$c + x \sin\theta,$$

and the pressure on an elementary strip

$$= g\rho\,(c + x\sin\theta)\, b\delta x;$$

$$\therefore\ \bar{x} = \frac{\int_0^a x\,(c + x\sin\theta)\,dx}{\int_0^a (c + x\sin\theta)\,dx} = \frac{a}{3}\,\frac{3c + 2a\sin\theta}{2c + a\sin\theta},$$

and the value of $\bar{y}$ is evidently zero.

If $\theta = 0$, $\bar{x} = \dfrac{a}{2}$, but if $c = 0$, $\bar{x} = \frac{2}{3}a$,

results illustrative of the remarks of Art. 38.

Ex. 3. *A quadrant of a circle just immersed vertically in a heavy homogeneous liquid, with one edge in the surface.*

Take Ox, the edge in the surface, as the axis of x,

then $p = g\rho y$,

and $$\bar{x} = \frac{\int_0^a\int_0^{\sqrt{(a^2-x^2)}} xy\,dx\,dy}{\int_0^a\int_0^{\sqrt{(a^2-x^2)}} y\,dx\,dy},\quad \bar{y} = \frac{\iint y^2\,dxdy}{\iint y\,dxdy},$$

the limits of the integrations for $\bar{y}$ being the same as for $\bar{x}$.

$$\iint y\,dxdy = \tfrac{1}{2}\int (a^2 - x^2)\,dx = \tfrac{1}{3}a^3,$$

$$\iint xy\,dxdy = \tfrac{1}{2}\int x\,.\,(a^2 - x^2)\,dx = \tfrac{1}{8}a^4,$$

$$\iint y^2\,dxdy = \tfrac{1}{3}\int (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{\pi a^4}{16};$$

$$\therefore\ \bar{x} = \frac{3}{8}\,a,\quad \bar{y} = \frac{3}{16}\,\pi a.$$

Employing polar co-ordinates and taking the line Ox as the initial line, we should have $p = g\rho r \sin\theta$, and

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}}\int_0^a r^3 \cos\theta \sin\theta \, dr \, d\theta}{\iint r^2 \sin\theta \, dr \, d\theta} = \frac{3}{8} a,$$

$$\text{and } \bar{y} = \frac{\int_0^{\frac{\pi}{2}}\int_0^a r^3 \sin^2\theta \, dr \, d\theta}{\iint r^2 \sin\theta \, dr \, d\theta} = \frac{3}{16} \pi a.$$

Ex. 4. *A circular area, radius* a, *is immersed with its plane vertical, and its centre at a depth* c.

Take the centre as the origin, and the vertical downwards from the centre as the initial line; then if p be the pressure at the point r, θ,

$$p = g\rho \,(c + r\cos\theta),$$

and the depth below the centre of the centre of pressure

$$= \frac{2\int_0^{\pi}\int_0^a r^2 \cos\theta \,(c + r\cos\theta)\, dr \, d\theta}{2\iint r\,(c + r\cos\theta)\, dr \, d\theta} = \frac{a^2}{4c}.$$

Ex. 5. *A vertical rectangle, exposed to the action of the atmosphere at a constant temperature.*

If Π be the atmospheric pressure at the base of the rectangle, the pressure at a height z is $\Pi\epsilon^{-\frac{gz}{k}}$, Art. (28), and if b denote the breadth, the pressure upon a horizontal strip of the rectangle

$$= \Pi\epsilon^{-\frac{gz}{k}} b\delta z,$$

$\therefore$ the resultant pressure, if a be the height,

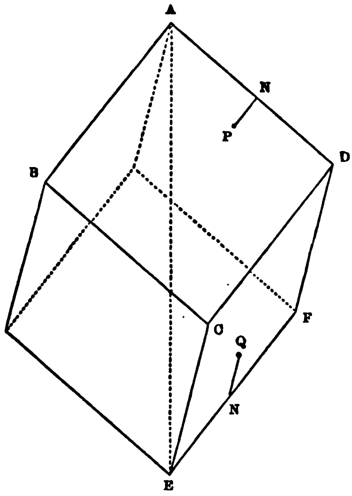
$$= \int_0^a \Pi\epsilon^{-\frac{gz}{k}} b dz = \Pi \frac{bk}{g} \left(1 - \epsilon^{-\frac{ga}{k}}\right),$$

and the height of the centre of pressure

$$=\frac{\int_0^a ze^{-\frac{gz}{k}}dz}{\int_0^a \epsilon^{-\frac{gz}{k}}dz}=\frac{k}{g}-\frac{a}{\epsilon^{\frac{ga}{k}}-1}.$$

Ex. 6. *A hollow cube is very nearly filled with liquid, and rotates uniformly about a diagonal which is vertical; required to find the pressures upon, and the centres of pressure of, its several faces.*

I. For one of the upper faces $ABCD$,
take AD, AB, as axes of x and y; z, r, the vertical and horizontal distances of any point $P(x, y)$ from A,



then $\frac{p}{\rho}=\frac{1}{2}\omega^2r^2+gz,$

$z=\frac{x+y}{\sqrt{3}}$, projecting the broken line ANP on AE,

$r^2=AP^2-z^2=x^2+y^2-z^2=\frac{2}{3}(x^2+y^2-xy)$;

$\therefore$ the pressure (P) on $ABCD=\int_0^a\int pdy\,dx$

$$=\rho\cdot\iint\left\{\frac{\omega^2}{3}(x^2+y^2-xy)+\frac{g}{\sqrt{3}}(x+y)\right\}dydx$$

$$=\rho\left\{\frac{5}{36}a^4\omega^2+\frac{g}{\sqrt{3}}a^3\right\}.$$

The centre of pressure is given by the equations

$$\bar{x}P = \bar{y}P = \rho\int_0^a\int_0^a x\left\{\frac{\omega^2}{3}(x^2+y^2-xy)+\frac{g}{\sqrt{3}}(x+y)\right\}dy\,dx;$$

$$\therefore \bar{x}=\bar{y}=a.\frac{21g+3\sqrt{3}\omega^2 a}{36g+5\sqrt{3}\omega^2 a}.$$

II. For one of the lower faces $ECDF$, take EF, EC as axes, then, for a point Q,

$$z = a\sqrt{3}-\frac{x+y}{\sqrt{3}},$$

$$r^2=\tfrac{4}{3}(x^2+y^2-xy),$$

and the rest of the process is the same as in the first case.

Ex. 7. *A quadrant of a circle is just immersed vertically, with one edge in the surface, in a liquid, the density of which varies as the depth.*

Taking Ox as the edge, in the surface, $\rho=\mu y$ and $p=\frac{1}{2}\mu g y^2$; the centre of pressure is therefore given by the equations

$$\bar{x}=\frac{\int_0^a\int_0^{\sqrt{a^2-x^2}} xy^2\,dy\,dx}{\iint y^2\,dy\,dx}, \text{ and } \bar{y}=\frac{\iint y^3\,dy\,dx}{\iint y^2\,dy\,dx};$$

or, in polar co-ordinates,

$$\bar{x}=\frac{\int_0^{\frac{\pi}{2}}\int_0^a r^4\sin^2\theta\cos\theta\,dr\,d\theta}{\iint r^3\sin^2\theta\,dr\,d\theta}, \text{ and } \bar{y}=\frac{\iint r^4\sin^3\theta\,dr\,d\theta}{\iint r^3\sin^2\theta\,dr\,d\theta};$$

and it will be found that

$$\bar{x}=\frac{16}{15}\frac{a}{\pi} \text{ and } \bar{y}=\frac{32}{15}\frac{a}{\pi}.$$

38. *A vessel having a plane base and plane vertical sides, contains two liquids which do not mix; to find the resultant pressure on one of the sides, and the centre of pressure.*

Let ρ be the density and h the depth of the upper liquid, ρ', h', corresponding quantities for the lower liquid; the common surface

must be a horizontal plane, (Art. 26), the pressure at any point of which is $g\rho h$, and the pressure at a depth z below the *common* surface is $g\rho h + g\rho' z$.

Taking b for the breadth of one of the vertical sides, the pressure of the upper liquid upon it $= \frac{1}{2} g\rho b h^2$, and the pressure of the lower liquid

$$= \int_0^{h'} g(\rho h + \rho' z)\, b dz = gbh' (\rho h + \tfrac{1}{2} \rho' h').$$

The resultant pressure is the sum of these two and is equal to

$$gb (\tfrac{1}{2}\rho h^2 + \rho h h' + \tfrac{1}{2}\rho' h'^2).$$

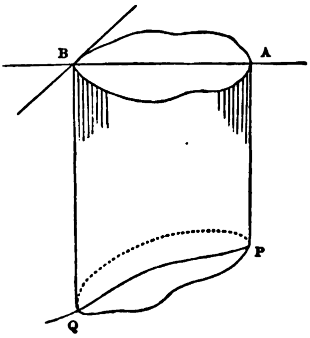
The moment of the fluid pressure on this side about its line of intersection with the surface

$$= \int_0^h g\rho b z^2 dz + \int_0^{h'} g(\rho h + \rho' z)\, b\, (h + z)\, dz:$$

performing the integrations, and dividing by the expression for the resultant pressure investigated above, we obtain the depth of the centre of pressure.

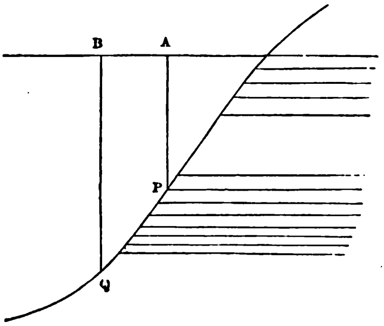
39. *To find the resultant vertical pressure on any surface of a homogeneous liquid at rest under the action of gravity.*

Let PQ be a surface exposed to the action of a heavy liquid; let AB be the projection of PQ on the surface of the liquid, and suppose the portion contained between PQ and the vertical lines through its boundary which meet the surface in AB to be solidified.



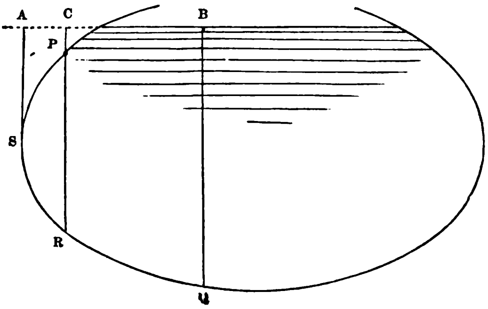
The solid AQ is supported by the horizontal pressure of the liquid and by the reaction of PQ; this reaction resolved vertically must be equal to the weight of AQ, and conversely, the pressure on PQ is equal to the weight of AQ, and acts through its centre of gravity.

If PQ be pressed upwards by the liquid as in the figure, produce the surface, project PQ on it as before, suppose the space AQ to be filled with liquid of the same kind, and remove the liquid from the inside.



Then the pressures at all points of PQ are the same as before, but in the contrary direction, and since the vertical pressure in this hypothetical case is equal to the weight of AQ, it follows that in the actual case, the resultant vertical pressure upwards is equal to the weight of AQ.

If the surface be pressed partially upwards and partially downwards, draw through P, the highest point of the portion of surface considered, a vertical plane PR, and let ACB be the projection of PSQ on the surface of the liquid.



Then the resultant vertical pressure on PSR,

$$= \text{the weight of the liquid in } PSR,$$

and on $RQ = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ CQ,$

and the whole vertical pressure = the weight of the liquid in CQ + the weight of the liquid in PR.

This might also have been deduced from the two previous articles, for PR can be divided by the line of contact of vertical tangent planes into two portions PS, SR, on which the pressures are respectively upwards and downwards; and since

pressure on PS = weight of liquid APS,

and............... SR = ... ASR,

the difference of these, i.e. the vertical pressure on PR = weight of fluid PR.

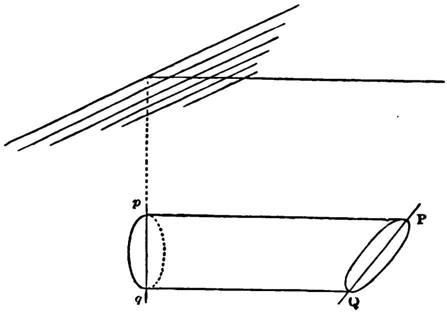
In a similar manner other cases may be discussed.

It will be observed that this investigation applies also to the case of a heterogeneous liquid (in which the density must be a function of the depth, since surfaces of equal pressure are surfaces of equal density), provided we consider that the hypothetical extension of the liquid follows the same law of density.

40. *To find the resultant horizontal pressure, in a given direction, on a surface* PQ.

Project PQ on a vertical plane perpendicular to the given direction, and let pq be the projection.

Then supposing Pq solidified, it is kept at rest by the pressure on pq, the resultant horizontal pressure on PQ, and forces in vertical planes parallel to the plane pq.



Hence the horizontal pressure on PQ is equal to that on pq, and acts in the same straight line, i.e. through the centre of pressure of pq.

41. Hence, in general, to determine the resultant fluid pressure on any surface, find the vertical pressure, and the resultant horizontal pressures in two directions at right angles to each other. These three forces may in some cases be compounded into a single force, the condition for which may be determined by the usual methods of Statics.

Ex. *A hemisphere is filled with homogeneous liquid: required to find the resultant action on one of the four portions into which it is divided by two vertical planes through its centre at right angles to each other.*

Taking the centre O as origin, the bounding horizontal radii as axes of x and y, and the vertical radius as the axis of z, the pressure parallel to x is equal to the pressure on the quadrant yOz, which is the projection, on a plane perpendicular to Ox, of the curved surface.

Therefore, the pressure parallel to Ox

$$= g\rho \frac{\pi a^2}{4} \cdot \frac{4a}{3\pi} = \frac{1}{3} g\rho a^3,$$

and the co-ordinates of its point of action are

$$\left(o, \ \frac{3}{8}a, \frac{3}{16}\pi a\right), \quad \text{Art. 39, Ex. 3};$$

similarly, the pressure parallel to $Oy = \frac{1}{3} g\rho a^3$, and acts through the point,

$$\left(\frac{3}{8}a, \quad o, \quad \frac{3}{16}\pi a\right).$$

The resultant vertical pressure = the weight of the liquid $= \frac{1}{6} g\rho\pi a^3$, and acts in the direction of the line $x = \frac{3}{8} a = y$.

The directions of the three forces all pass through the point

$$\left(\frac{3}{8}a, \ \frac{3}{8}a, \ \frac{3}{16}\pi a\right),$$

and they are therefore equivalent to a single force

$$\frac{1}{6} g\rho a^3 \sqrt{(\pi^2 + 8)} \text{ in the line}$$

$$x-\frac{3}{8}a=y-\frac{3}{8}a\doteq\frac{2}{\pi}\left(z-\frac{3}{16}\pi a\right),$$

$$\text{or}\quad x=y=\frac{2}{\pi}z,$$

a straight line through the centre, as must obviously be the case, since all the fluid pressures are normal to the surface. The point in which it meets the surface of the hemisphere may be called 'the centre of pressure.'

42. *To find the resultant pressure on the surface of a solid either wholly or partially immersed in a heavy liquid.*

Suppose the solid removed, and the space it occupied filled with liquid of the same kind, and conceive this liquid solidified; the resultant pressure upon it will be the same as upon the original solid. But the solidified mass is at rest under the action of its own weight, and the pressure of the liquid surrounding it: the resultant pressure is therefore equal to the weight of the liquid displaced, and acts in a vertical line through its centre of gravity*.

The same reasoning evidently shews that the resultant pressure of an elastic fluid on any solid is equal to the weight of the elastic fluid displaced by the solid.

43. *To find the resultant pressure on any surface of a fluid at rest under the action of any given forces.*

Let p be the pressure, determined as in Chapter II., at any point (x, y, z) of a surface, $u=0$, exposed to the action of the fluid. Then if

$$\frac{1}{P^2}=\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2,$$

* This result may also be obtained by means of Arts. 41 and 42, as follows: Draw parallel horizontal lines touching the surface, and forming a cylinder which encloses it; the curve of contact divides the surface into two parts, on which the resultant horizontal pressures, parallel to the axis of the cylinder, are by Art. 42 equal and opposite; the horizontal pressures on the solid therefore balance each other and the resultant is wholly vertical. To determine the amount of the resultant vertical pressure, draw parallel vertical lines touching the surface, and dividing it into two portions on one of which the resultant vertical pressure acts upwards, and on the other downwards; the difference of the two, by Art. 41, is evidently the weight of the fluid displaced by the solid.

$$P\frac{du}{dx},\ P\frac{du}{dy},\ P\frac{du}{dz},$$

are the direction-cosines of the normal at the point (x, y, z).

Let δS be an element of the surface about the same point. The pressures on this element, parallel to the axes, are

$$pP\frac{du}{dx}\delta S,\qquad pP\frac{du}{dy}\delta S,\qquad pP\frac{du}{dz}\delta S;$$

$\therefore$ if X, Y, Z, and L, M, N, be the resultant pressures parallel to the axes, and the resultant couples, respectively,

$$X=\iint pP\frac{du}{dx}dS,\quad Y=\iint pP\frac{du}{dy}dS,\quad Z=\iint pP\frac{du}{dz}dS,$$

$$L=\iint pPdS\left(y\frac{du}{dz}-z\frac{du}{dy}\right),$$

$$M=\iint pPdS\left(z\frac{du}{dx}-x\frac{du}{dz}\right),$$

$$N=\iint pPdS\left(x\frac{du}{dy}-y\frac{du}{dx}\right);$$

the integrations being made to include the whole of the surface under consideration.

These resultants are equal to a single force if

$$XL+YM+ZN=0.$$

44. The surface may be divided into elements in three different ways by planes parallel to the co-ordinate planes.

Thus, $\delta x\delta y =$ projection of δS on $xy = P\frac{du}{dz}\delta S$;
and $\therefore Z=\iint pdxdy$; and similarly, $X=\iint pdydz$, and $Y=\iint pdzdx$,

$$L=\iint p\,(ydxdy-zdzdx),$$

$$=\iint p\,(ydy-zdz)\,dx,$$

$$M=\iint p\,(zdz-xdx)\,dy,$$

$$N=\iint p\,(xdx-ydy)\,dz.$$

45. If the fluid be at rest under the action of gravity only, and the axis of z be vertical, p is a function of z, $\phi(z)$ suppose, and therefore,

$$X = \iint \phi(z)\, dydz,$$

which is evidently the expression for the pressure, parallel to x, upon the projection of the given surface on the plane yz; and similarly Y is equal to the pressure upon the projection on xz.

Again, if the fluid be incompressible and acted upon by gravity only, $p\delta x\delta y$ is equal to the weight of the portion of fluid contained between δS and its projection on the surface of the fluid;

$\therefore Z$, or $\iint pdxdy$, is the weight of the superincumbent fluid.

These results accord with those previously obtained, Arts. 39 and 40.

46. If a solid body be wholly or partially immersed in any fluid which is at rest under the action of given forces, the resultant fluid pressure on the body will be equal to the resultant of the forces which would act on the displaced fluid.

For we can imagine the solid removed and the gap filled up with the fluid, which will be in equilibrium under the action of the forces and the pressure of the surrounding fluid; and the resultant pressure must be equal and opposite to the resultant of the forces.

In filling up the gap with fluid, the law of density must be maintained, that is, the surfaces of equal density must be continuous with those of the surrounding fluid.

EXAMPLES.

1. Find the centre of pressure of a parallelogram with one side in the surface, and of a triangle with one side in the surface.

2. Water is poured into a hollow sphere, determine the depth of the water when the resultant pressure is half the total normal pressure.

3. A conical wine-glass is filled with water and placed in an inverted position on a table; if the whole pressure of the water on the glass be double its resultant pressure, find the vertical angle of the cone.

4. A hollow paraboloidal vessel, open at the top, is inverted and placed on a horizontal table; fluid being poured in through a hole at the vertex, find its height when it begins to escape, and the condition that this may be possible.

5. A vessel in the form of a regular pyramid, whose base is a plane polygon of n sides, is placed with its axis vertical and vertex downwards and is filled with fluid. Each side of the vessel is moveable about a hinge at the vertex, and is kept in its place by a string fastened to the middle point of its base and to the centre of the polygon: shew that the tension of each string is to the whole weight of the fluid as 1 to $n \sin 2\alpha$, where α is the inclination of each side to the horizon.

6. Find the centre of pressure of a square lamina having one angular point in the surface of a liquid; and supposing it to be moved about the angular point in its own plane, which is fixed, and to be always totally immersed, find the locus on its own plane of its centre of pressure.

7. Find the centre of pressure of an elliptic lamina just immersed in water; and supposing it turned round in the same vertical plane, so as to be always just immersed, find the locus with respect to its axes of the centre of pressure.

8. A cubical box, filled with water, has a close-fitting heavy lid fixed by smooth hinges to one edge; compare the tangents of the angles through which the box must be tilted about the several edges of its base, in order that the water may just begin to escape.

9. A plane area of any form is immersed in a homogeneous liquid, and vertical straight lines are drawn through the boundary to the surface, thus determining a portion of the liquid: shew that the centre of gravity of this portion and the centre of pressure of the plane area are in the same vertical straight line, and that the depth of the centre of pressure is twice that of the centre of gravity.

10. Find the centre of pressure of a semi-ellipse (axes $2a$ and a) which is bounded by a diameter inclined at the angle $\frac{\pi}{6}$ to its major axis, its plane being vertical, and the diameter in the surface.

11. A semi-ellipse bounded by its axis minor, is just immersed in a liquid the density of which varies as the depth; if the axis minor be in the surface, find the eccentricity in order that the focus may be the centre of pressure.

12. A square lamina $ABCD$, which is immersed in water, has the side AB in the surface; draw a line BE to a point E in CD such that the pressures on the two portions may be equal. Prove that, if this be the case, the distance between the centres of pressure : the side of the square :: $\sqrt{505}$: 48.

13. From a semicircle, whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semicircle; find the centre of pressure of the remainder.

14. A semicircular lamina is completely immersed in water with its plane vertical, so that the extremity A of its bounding diameter is in the surface, and the diameter makes with the surface an angle α.

Prove that if E be the centre of pressure and θ the angle between AE and the diameter,

$$\tan\theta = \frac{3\pi + 16\tan\alpha}{16 + 15\pi\tan\alpha}.$$

15. Find the centre of pressure of a segment of a parabola bounded by the curve and the latus-rectum, the tangent at one end of the bounding ordinate being in the surface. If the liquid rise, the parabola remaining stationary, shew that the centre of pressure describes a straight line.

16. A cone is totally immersed in water, the depth of the centre of its base being given. Prove that, P, P', P'', being the resultant pressures on its convex surface, when the sines of the inclination of its axis to the horizon are s, s', s'', respectively,

$$P^2(s'-s'')+P'^2(s''-s)+P''^2(s-s')=0.$$

17. Find the centre of pressure of the area between the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, and the axes, taking the axes rectangular and one of them in the surface.

18. A quantity of liquid acted upon by a central force varying as the distance is contained between two parallel planes; if A, B, be the areas of the planes in contact with the fluid, shew that the pressures upon them are in the ratio $A^2 : B^2$.

19. A hollow sphere is full of liquid, the density of which varies as (the depth)^n; shew that the whole pressure on the surface of the sphere : the resultant pressure :: $n+3 : n+1$.

20. One asymptote of a hyperbola lies in the surface of a fluid; find the depth of the centre of pressure of the area included between the immersed asymptote, the curve, and two given horizontal lines in the plane of the hyperbola.

21. A cone is immersed in water with the centre of its base at a distance of $\frac{5}{6}$ of its altitude below the surface. A paraboloid of the same base and altitude is also immersed with the centre of its base at the same distance below the surface as that of the cone, and with its axis inclined at the same angle to the vertical. Find what this angle must be in order that the resultant pressures on the convex surfaces of the two solids may be equal.

22. A hollow cube is very nearly filled with liquid, and is made to rotate uniformly about a vertical edge; find the pressure upon, and centres of pressure of, its several sides.

23. A closed cylinder, very nearly filled with liquid, rotates uniformly about a generating line, which is vertical; find the resultant pressure on its curved surface.

Determine also the point of action of the pressure on its upper end.

24. Shew that the depth of the centre of pressure of the area included between the arc and the asymptote of the curve $(r-a)\cos\theta = b$ is $\frac{a}{4} \cdot \frac{3\pi a + 16b}{3\pi b + 4a}$, the asymptote being in the surface and the plane of the curve vertical.

25. A cone is filled with liquid, and fitted with a heavy lid, moveable about a hinge; it is then made to revolve uniformly about the generating line through the hinge, which is vertical; find the greatest angular velocity consistent with no escape of the liquid.

26. A portion of a spherical shell is cut off by a plane, and the remaining portion is placed on a horizontal plane so that the circular section is in contact with the plane and is then filled with water through a small hole at the highest point. Find the largest piece which can be cut off so that, however light the shell may be, the water may not escape.

In this case, prove that the whole pressure on the shell is to the weight of the liquid in the ratio 2 : 1.

27. If a plane area immersed in a liquid revolve about any axis in its own plane, prove that the centre of pressure describes a straight line in the plane.

28. A plane area is wholly immersed in a liquid in a position not horizontal; if the area be turned about its centre of gravity in its own plane, shew that the locus of the centre of pressure of the area will be an ellipse.

29. A cube whose edge is $2a$, and whose faces are horizontal and vertical, is surrounded by a mass of heavy liquid, the volume of which is $8a^3\{\pi\sqrt{6}-1\}$; the liquid is acted on by a force tending to the centre of the cube, and varying as the distance, the force at the

distance a being g: find the form of the free surface and the pressure at any point: also if one of the vertical faces of the cube be moveable about a horizontal line in its own plane, shew that the face will be at rest, if this line be at a distance $\frac{4}{5}a$ from the lowest edge of that face.

30. A conical vessel, axis vertical and vertex downwards, is divided into two parts by a plane through its axis, and the parts are prevented from separating by a string which is a diameter of the rim of the vessel, and is perpendicular to the dividing plane, and by a hinge at the vertex.

The vessel being filled with water, compare the tension of the string with the weight of the water.

31. A hollow cone open at the top is filled with water; find the resultant pressure on the portion of its surface cut off, on one side, by two planes through its axis inclined at a given angle to each other; also determine the line of action of the resultant pressure, and shew that, if the vertical angle be a right angle, it will pass through the centre of the top of the cone.

32. A solid cylinder having plane ends is completely immersed in water with its centre at a given depth and its axis inclined at a given angle to the vertical; find the direction and magnitude of the resultant pressure on the curved surface of the cylinder.

33. A bowl in the form of a hemisphere is filled with water; find the direction and magnitude of the resultant pressure on the upper portion of the bowl cut off by a plane through its centre inclined at a given angle to the horizon.

34. An open conical shell, the weight of which may be neglected, is filled with water, and is then suspended from a point in the rim, and allowed gradually to take its position of equilibrium; prove that, if the vertical angle be $\cos^{-1}\frac{2}{3}$, the surface of the water will divide the generating line through the point of suspension in the ratio 2 : 1.

35. A solid is formed by turning a circular area round a tangent line through an angle α, and this solid is held under water, totally immersed, with its lower plane face horizontal, and at a given depth; find the direction and magnitude of the resultant pressure on the curved surface.

36. A regular polygon wholly immersed in a liquid is moveable about its centre of gravity; prove that the locus of the centre of pressure is a sphere.

37. A hemispherical bowl is filled with water, and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface; if 2α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical

$$= \tan^{-1}\left(\frac{\sin \alpha}{\alpha}\right).$$

38. A vessel in the form of a surface of revolution has the following property; if it be placed with its axis vertical, and any quantity of water be poured into it, the ratio of the total normal pressure to the resultant vertical pressure varies as the depth of the water poured in. Shew that the equation to the generating curve is

$$cs = xy.$$

CHAPTER IV.

THE EQUILIBRIUM OF FLOATING BODIES.

47. *To find the conditions of equilibrium of a floating body.*

We shall suppose that the fluid is at rest under the action of gravity only, and that the body, under the action of the same force, is floating freely in the fluid. The only forces then which act on the body are its weight, and the pressure of the surrounding fluid, and in order that equilibrium may exist, the resultant fluid pressure must be equal to the weight of the body, and must act in a vertical direction.

Now we have shewn, Art. (42), that the resultant pressure of a heavy fluid on the surface of a solid, either wholly or partially immersed, is equal to the weight of the fluid displaced, and acts in a vertical line through its centre of gravity.

Hence it follows that the weight of the body must be equal to the weight of the fluid displaced, and that the centres of gravity of the body, and of the fluid displaced, must lie in the same vertical line.

These conditions are necessary and sufficient conditions of equilibrium, whatever be the nature of the fluid in which the body is floating. If it be heterogeneous, the displaced fluid must be looked upon as following the same law of density as the surrounding fluid; in other words, it must consist of strata of the same kind as, and continuous with, the horizontal strata of uniform density, in which the particles of the surrounding fluid are necessarily arranged.

If for instance a solid body float in water, partially immersed, its weight will be equal to the weight of the water

displaced, together with the weight of the air displaced; and if the air be removed, or its pressure diminished by a diminution of its density or temperature, Art. (18), the solid will sink in the water through a space depending upon its own weight, and upon the densities of air and water. This may be further explained by observing that the pressure of the air on the water is greater than at any point above it, and that this surface pressure of the air is transmitted by the water to the immersed portion of the floating body, and consequently the upward pressure of the air upon it is greater than the downward pressure.

48. We now proceed to illustrate the application of the above conditions, by the discussion of some particular cases.

Ex. 1. *A portion of a solid paraboloid, of given height, floats with its axis vertical and vertex downwards in a homogeneous liquid, required to find its position of equilibrium.*

Taking $4a$ as the latus rectum of the generating parabola, h its height, and x the depth of its vertex, the volumes of the whole solid and of the portion immersed are respectively $2\pi ah^2$ and $2\pi ax^2$; and if ρ, σ, be the densities of the solid and liquid, one condition of equilibrium is

$$\rho \,.\, 2\pi ah^2 = \sigma \,.\, 2\pi ax^2;$$

$$\therefore\ x = \sqrt{\frac{\rho}{\sigma}}\,h,$$

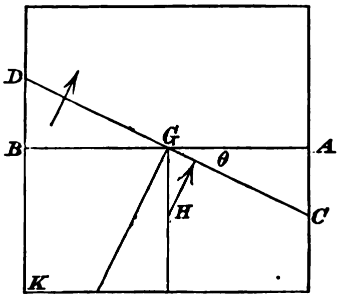
which determines the portion immersed, the other condition being obviously satisfied.

Ex. 2. *It is required to find the positions of equilibrium of a square lamina floating with its plane vertical, in a liquid of double its own density.*

The conditions of equilibrium are clearly satisfied if the lamina float half immersed either with a diagonal vertical, or with two sides vertical.

To examine whether there is any other position of equilibrium, let the lamina be held with the line DGC in the

surface, in which case the first condition is satisfied.



But, if the angle $CGA = \theta$, and if $2a$ be the side of the square, the moment about G of the fluid pressure, which is the same as the difference between the moments of the rectangle AK, and of twice the triangle GBD,

$$\propto 2a^2 \cdot \frac{a}{2} \sin\theta - a^2 \tan\theta \cdot \frac{a\sec\theta + a\cos\theta}{3},$$

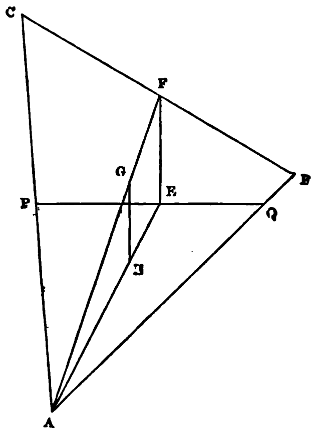
$$\propto \sin\theta\,(1 - \tan^2\theta);$$

and this vanishes only when $\theta = 0$ or $\frac{\pi}{4}$.

Hence there is no other position of equilibrium.

Ex. 3. *A triangular prism floats with its edges horizontal, to find its positions of equilibrium.*

Let the figure be a section of the prism by a vertical plane through its centre of gravity.



PQ is the line of floatation and H the centre of gravity of the liquid displaced. When there is equilibrium the area APQ is to ABC in the ratio of the density of the prism to the density of the liquid, and therefore for all possible positions of PQ the area APQ is constant; hence PQ always touches, at its middle point, an hyperbola of which AB, AC, are the asymptotes.

Also HG must be perpendicular to PQ, and therefore since

$$AH : HE = AG : GF,$$

FE must be perpendicular to PQ, that is, FE is the normal at E to the hyperbola. The problem is therefore reduced to that of drawing normals from F to the curve.

Let $xy = c^2$ be the equation of the curve referred to AB, AC as axes, and let

$$\angle BAC = \theta,\ AB = 2a,\ AC = 2b. \quad \ldots\ldots\ldots.(\alpha).$$

Let x, y, be the co-ordinates of E; the co-ordinates of F are a, b, and the equation of the normal at E is

$$\eta - y = \frac{y \cos\theta - x}{x \cos\theta - y} (\xi - x).$$

And if this pass through F, the co-ordinates of which are a, b,

$$(b - y)(x \cos\theta - y) = (a - x)(y \cos\theta - x),$$

$$\text{or } x^2 - (a + b\cos\theta)\, x = y^2 - (a\cos\theta + b)\, y \ldots\ldots(\beta).$$

The equations (α) and (β) determine all the points of the hyperbola, the tangents at which can be lines of floatation.

Also (β) is the equation to an equilateral hyperbola, referred to conjugate diameters parallel to AB, AC; the points of intersection of the two hyperbolas are therefore the positions of E.

To find x, we have

$$x^4 - (a + b\cos\theta) \,.\, x^3 + (a\cos\theta + b)\, c^2 x - c^4 = 0,$$

an equation which has only one negative root, and one or three positive roots, and there may be therefore three positions of equilibrium or only one.

If the densities of the liquid and the prism be ρ and σ, we have, since the area PAQ

$$=\tfrac{1}{2}AP\,.\,AQ\sin\theta = 2xy\sin\theta = 2c^2\sin\theta,$$

$$2\rho c^2\sin\theta = 2\sigma ab\sin\theta,$$

$$\text{or}\quad \rho c^2 = \sigma ab,$$

from which c is determined.

Suppose the prism to be isosceles, then putting $a=b$, the equation for x becomes

$$x^4 - c^4 - a(1+\cos\theta)(x^3 - c^2x) = 0;$$

from which we obtain $x=c$, which gives $y=c$, and makes BC horizontal, an obvious position of equilibrium, and also

$$x = \frac{a}{2}(1+\cos\theta) \pm \left\{\frac{a^2}{4}(1+\cos\theta)^2 - c^2\right\}^{\frac{1}{2}}$$

$$= a\cos^2\frac{\theta}{2} \pm \left(a^2\cos^4\frac{\theta}{2} - c^2\right)^{\frac{1}{2}};$$

the isosceles prism will therefore have only one position of equilibrium, unless

$$a\cos^2\frac{\theta}{2} > c;$$

and, since $\rho c^2 = \sigma a^2$, this is equivalent to

$$\cos^2\frac{\theta}{2} > \sqrt{\frac{\sigma}{\rho}}.$$

Ex. 4. *Determine the position of equilibrium of a balloon of given size and weight, neglecting the variations of temperature at different heights in the atmosphere.*

If the temperature be constant, the pressure of the air at a height $z = \Pi\epsilon^{-\frac{gz}{k}}$, and its density $= \frac{\Pi}{k}\epsilon^{-\frac{gz}{k}}$, Π being the atmospheric pressure at the level from which the height is measured.

The air displaced consists of a series of strata of variable density, and if z be the height of the lowest point of the balloon, x the distance from that point of any horizontal section (X) of the balloon, and h its height, the weight of a stratum of the air displaced is

$$\frac{\Pi g}{k} \epsilon^{-\frac{g(z+x)}{k}} X\delta x,$$

and the whole weight of air displaced

$$= \int_0^h \frac{\Pi g}{k} \epsilon^{-\frac{g(z+x)}{k}} X dx = \frac{\Pi g}{k} \epsilon^{-\frac{gz}{k}} \int_0^h \epsilon^{-\frac{gx}{k}} X dx.$$

The form of the balloon being given, X is a known function of x, and if W be the weight of the balloon and of the gas it contains, the height z will be determined by equating W to the expression we have obtained for the weight of the air displaced.

49. *A homogeneous solid floats, wholly immersed, in a liquid of which the density varies as the depth; to find the depth of its centre of gravity.*

Let a, c, be the depths of the highest and lowest points of the solid, Z the area of a horizontal section of the solid at a depth z, and μz the density;

$$\text{the weight of the liquid displaced} = \int_a^c g\mu z Z dz.$$

Let $\bar{z}$ be the depth of the centre of gravity of the solid, and V its volume, then

$$V\bar{z} = \int_a^c Zz dz;$$

therefore the weight of displaced liquid $= g\mu\bar{z}V$, and if ρ be the density of the solid, its weight $= g\rho V$; hence $\rho = \mu\bar{z}$, or the solid floats in such a position that the density of the liquid at the depth of the centre of gravity of the solid is equal to the density of the solid.

50. If a solid float *under constraint*, the conditions of equilibrium depend on the nature of the constraining circumstances,

but in any case the resultant of the constraining forces must act in a vertical direction, since the other forces, the weight of the body, and the fluid pressure, are vertical.

If for instance one point of a solid be fixed, the condition of equilibrium is that the weight of the body and the weight of the fluid displaced should have equal moments about the fixed point; this condition being satisfied, the solid will be at rest, and the stress on the fixed point will be the difference of the two weights.

As an additional illustration, consider the case of a solid floating in water and supported by a string fastened to a point above the surface; in the position of equilibrium the string will be vertical, and the tension of the string, together with the resultant fluid pressure, which is equal to the weight of the displaced fluid, will counterbalance the weight of the body; the tension is therefore equal to the difference of the weights, and the weights are inversely in the ratio of the distances of their lines of action from the line of the string, these three lines being in the same vertical plane.

51. For subsequent investigations, the following geometrical propositions will be found important.

If a solid be cut by a plane, and this plane be made to turn through a very small angle about a straight line in itself, the volume cut off will remain the same, provided the straight line pass through the centre of gravity of the area of the plane section.

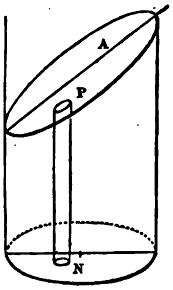
To prove this, consider a right cylinder of any kind cut by a plane making with its base an angle θ.

Let $\bar{z}$ be the distance from the base of the centre of gravity of the section A, δA an element of the area of the section and V the volume between the planes. Then

$$\bar{z} = \frac{\Sigma\,(\delta A\,.\,PN)}{A};$$

$$\therefore A \cos\theta \bar{z} = \Sigma\,(\delta A \cos\theta \,.\, PN) = V,$$

$$\text{or } V = \bar{z} \text{ (area of base)}.$$



Now the centre of gravity of the area A is also the centre of gravity of all sections made by planes passing through it, as may be seen by projecting the sections on the base of the cylinder; it follows therefore, that, $\bar{z}$ being the same for all such sections, the volumes cut off are the same.

In the case of any solid, if the cutting plane be turned through a very small angle about the centre of gravity of its section, the surface near the curves of section may be considered, without sensible error, cylindrical, and the above proposition is therefore established*.

In other words, the difference between the volume lost and the volume gained by the change in the position of the cutting plane will be indefinitely small compared with either.

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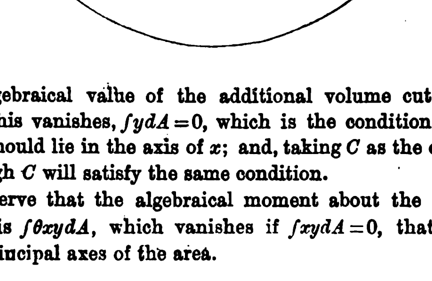
* The following form of proof may also be given.

Let ACB, the cutting plane, be turned through a small angle (θ) about a line Cx, and let dA be an element of the area.

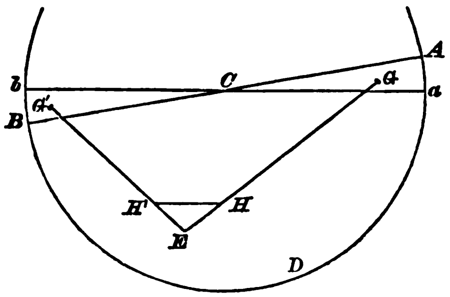


Then the algebraical value of the additional volume cut off is equal to $\int \theta y dA$, and, if this vanishes, $\int y dA = 0$, which is the condition that the centre of gravity of A should lie in the axis of x; and, taking C as the centre of gravity, any plane through C will satisfy the same condition.

We may observe that the algebraical moment about the axis of y of the volume cut off is $\int \theta xy dA$, which vanishes if $\int xy dA = 0$, that is, if the axes Cx, Cy be the principal axes of the area.

52. *If a plane move so as to cut from a solid a constant volume, and if H be the centre of gravity of the volume cut off, the tangent plane at H to the surface which is the locus of H is parallel to the cutting plane.*

Turn the plane ACB, the cutting plane, through a small



angle into the position aCb, the volumes of the wedges ACa, BCb being equal.

Let G and G' be the centres of gravity of these wedges.

In GH produced take a point E such that

$$EH : HG :: \text{Volume } ACa : \text{Volume } aDB.$$

Join EG' and take H' such that

$$EH' : H'G' :: \text{Vol. } BCb : \text{Vol. } aDB;$$

then H' is the centre of gravity of aDb;

$$\therefore EH : HG :: EH' : H'G',$$

and HH' is therefore parallel to GG'.

Hence it follows that ultimately when the angle ACa is indefinitely diminished,

$$HH' \text{ is parallel to } ACB;$$

and HH' is a tangent at H to the locus of H.

This being true for any displacement of the plane ACB about its centre of gravity, it follows that the tangent plane at H to the locus of H is parallel to the plane ACB.

53. *Definitions.* If a body float in a homogeneous liquid, the plane in which the body is intersected by the surface of the liquid is the *plane of floatation.*

The point H, the centre of gravity of the liquid displaced, is called the *Centre of Buoyancy.*

If the floating body be displaced in any manner, the centre of gravity, H, of the liquid displaced, is still called the *centre of buoyancy;* and, if the volume of liquid displaced remain unchanged, the locus of the point H is called the *Surface of Buoyancy.*

Considering a particular vertical plane of displacement through the line HG, the intersection of the surface of buoyancy by this plane is the *Curve of Buoyancy.*

Thus, in the case of Example (3), Art. (48), the curve of buoyancy (since $AH = \frac{2}{3} AE$), is an hyperbola similar to the hyperbola which is the envelope of the lines of floatation.

54. *The positions of equilibrium of a body floating in a homogeneous liquid are determined by drawing normals from G, the centre of gravity of the body, to the surface of buoyancy.*

For if GH be a normal to the surface of buoyancy, the tangent plane at H, being parallel to the plane of floatation, is horizontal, and GH is therefore vertical.

The two conditions of equilibrium are then satisfied, and a position of equilibrium is determined.

The problem comes to the same thing as determining the positions of equilibrium of a heavy body, bounded by the surface of buoyancy, on a horizontal plane.

55. *A solid of revolution floats in a liquid which rotates uniformly, as if solid, about a vertical axis, the axis of the solid coinciding with the axis of rotation; required to find the condition of equilibrium.*

In a mass of rotating liquid, suppose a surface of revolution described, having its axis coincident with the axis of rotation, and let the liquid within this surface be made solid. No immediate change of motion will be produced, and since the rotation is about a principal axis, and the fluid pressures on the solidified fluid are normal to its surface, no subsequent change will take place, and the solidified fluid will continue to rotate as before. The resultant of the fluid pressures upon this solid is therefore equal to its weight, and the same pressures being exerted on the surface of any solid occupying the same space, it follows that any such solid will be in equilibrium, if its weight be equal to the weight of the fluid it displaces.

It will be seen moreover that it is quite indifferent whether the solid rotate with the fluid, or with a different angular velocity, or be at rest.

Ex. *A cylinder floats in rotating liquid; to find the depth to which it is immersed.*

If ω be the angular velocity, the equation to the generating parabola of the free surface, taking its vertex as the origin, is $\omega^2 y^2 = 2gz$, and if z be the depth of the base of the cylinder below the circle of floatation, that is, the circle in which the free surface intersects the surface of the cylinder, and c the radius of the cylinder, the volume of the displaced fluid is the difference between the volume of a height z of the cylinder, and the volume of a height $\frac{\omega^2 c^2}{2g}$ of the paraboloid.

Hence, if σ be the density of the cylinder and ρ of the fluid,

$$\sigma \pi c^2 h = \rho \left(\pi c^2 z - \frac{\pi \omega^2 c^4}{4g}\right),$$

$$\text{and } z = \frac{\sigma}{\rho} h + \frac{\omega^2 c^2}{4g}.$$

56. A more general case is that of a body floating, wholly or partially immersed, in a liquid at rest under the action of any given forces, the same forces being supposed to act on the molecules of the body.

If the body be in equilibrium, the resulting force upon it will be equal to the resulting force on the liquid displaced, and the lines of action of the two forces will be the same.

For, if the body be removed, and its place occupied by the displaced liquid, the resulting pressure of the liquid upon the body will be the same as upon the displaced liquid, and will therefore be equal and opposite to the resultant force upon the displaced liquid.

Ex. *A mass of liquid is at rest under the action of a force to a fixed point varying as the distance, and a solid in the form of a spherical sector is at rest partly immersed in it, with its vertex at the fixed point; it is required to compare the densities of the liquid and the solid.*

In the state of equilibrium, let r be the radius of the free surface of the liquid, and a the radius of the spherical sector. The volumes of the sector and of the displaced liquid are in the ratio of a^3 to r^3; and the distances of their centre of gravity from the centre of force are in the ratio of a to r;

$$\therefore \text{ if } \rho \text{ and } \sigma \text{ be the densities, } \rho a^4 = \sigma r^4.$$

EXAMPLES.

1. A cylindrical block of wood is placed with its axis vertical in a cylindrical vessel whose base is plane, and water is then poured in to twice the height of the cylinder; find the pressure of the wood on the base of the vessel.

If the wood be displaced, and float, find the new pressure on that portion of the base with which it was previously in contact.

2. A cylinder floats between two liquids with its axis vertical, its height being equal to the depth of the upper liquid; compare the pressures on the two ends of the cylinder, the densities of the liquids and of the cylinder being given.

Is the equilibrium of the cylinder stable or unstable for a vertical displacement?

3. If a heterogeneous lamina, bounded by an ellipse, float with its plane vertical, prove that the problem of finding its positions of equilibrium is the same as that of drawing normals from the centre of gravity to a similar and concentric ellipse.

4. A cone, placed with its axis vertical and vertex downwards in a liquid, floats with half its axis immersed, and, when placed in another liquid, it floats with three-fourths immersed: in what proportion must these be mixed, that it may float in the mixture with two-thirds of the axis immersed?

5. A cone, of given weight and volume, floats with its vertex downwards; prove that the surface of the cone in contact with the liquid is least when its vertical angle is $2\tan^{-1}\dfrac{1}{\sqrt{2}}$.

6. A triangular lamina ABC, right-angled at C, is attached to a string at A, and rests with the side AC vertical and half its length immersed in liquid; prove that the density of the lamina is $\frac{7}{8}$ths of the density of the liquid.

7. A square board is placed in liquid of four times its density; shew that there are three different positions in which it will float with one given corner only below the surface of the fluid.

8. A body is floating in water; a hollow vessel is inverted over it and depressed: what effect will be produced in the position of the body, (1) with reference to the surface of the water within the vessel, (2) with reference to the surface of the fluid outside?

9. The base of a vessel containing water is a horizontal plane, and a sphere of less density than water is kept totally immersed by a string fastened to the middle point of a circular disc, which lies in contact with the base. Find the greatest sphere of given density, and also the sphere of given size and least density, which will not raise the disc.

Examine also, in each case, the effect of increasing the density of the liquid, or of diminishing its depth.

10. A hollow hemispherical shell has a heavy particle fixed to its rim, and floats in water with the particle just above the surface, and with the plane of the rim inclined at an angle of 45° to the surface; shew that the weight of the hemisphere : the weight of the water which it would contain

$$:: 4\sqrt{2}-5 : 6\sqrt{2}.$$

11. A thin hollow cone closed by an equally thin plane base will remain wherever it is placed entirely within a liquid: prove that its vertical angle is $2\sin^{-1}\cdot\dot{3}$.

12. A solid formed of two co-axial right cones, of the same vertical angle, connected at the vertices, is placed with one end in contact with the horizontal base of a vessel: water is then poured into the vessel; shew that if the altitude of the upper cone be treble that of the lower, and the common density of the spindle four-sevenths that of the water, it will be upon the point of rising when the water reaches to the level of its upper end.

13. A sphere of given radius floats in equilibrium in a quantity of water contained in a cylindrical vessel, revolving uniformly about its axis which is vertical; the velocity of rotation is such that the centrifugal acceleration at a distance from the axis equal to the radius

of the sphere is equivalent to the acceleration of gravity; prove that the whole pressure upon the sphere varies as the cube of the surface immersed.

14. A solid cone is divided into two parts by a plane through its axis, and the parts are connected by a hinge at the vertex; the system being placed in water with its axis vertical and vertex downwards, shew that, if it float without separation of the parts, the length of the axis immersed is greater than $h \sin^2 a$, h being the height of the cone, and $2a$ its vertical angle.

15. A cone, the vertex of which is fixed at the bottom of a vessel containing water, is in equilibrium, with its slant side vertical and the lowest point of its base just touching the surface. Compare the density of the cone with that of the water.

16. The curved surface of a cup is formed by the revolution of a portion of the curve $y = be^{\frac{x}{a}}$ about its asymptote. It floats in liquid with its axis vertical and narrow end downwards, and a heavier liquid is poured into it. Shew that if the cup be made of proper weight, the distance between the surfaces of the two liquids will be constant.

17. A cylinder floats in a liquid with its axis inclined at an angle $\tan^{-1}\frac{2}{5}$ to the vertical, and its upper end just above the surface; prove that the radius is $\frac{4}{7}$ of the height of the cylinder.

18. Two rods of the same substance have their ends fastened together, and float in a liquid with the angle immersed; shew that the problem of finding the positions of equilibrium is the same as that of drawing normals to a parabola from a given point within it.

19. A hollow hemispherical cup is closed by a lid of the same small thickness and of the same substance; shew that, if it float in a liquid with its centre in the surface, the inclination of the lid to the vertical will be $11^\circ 15'$.

20. A right circular cone has a plane base in the form of an ellipse; the cone floats with its longest generating line horizontal;

if $2a$ be the vertical angle, and β the angle between the plane base and the shortest generating line, shew that

$$\cot\beta = \cot 4a - \frac{1}{5}\operatorname{cosec} 4a.$$

21. If the height of a right circular cone be equal to the diameter of the base, it will float, with its slant side horizontal, in any liquid of greater density.

22. A cone, whose height is h and vertical angle $2a$, has its vertex fixed at distance c beneath the surface of a liquid; shew that it will rest with its base just out of the liquid if

$$\sigma c^4 . \cos^8 a . \cos\theta = \rho h^4 [\cos(\theta - a) . \cos(\theta + a)]^{\frac{5}{2}},$$

where σ and ρ are the densities of the liquid and cone, and θ is given by the equation $c \cos a = h \cos(\theta + a)$.

23. A tetrahedron floats in water with one corner immersed. The three edges which meet in this corner are equal and mutually at right angles. Shew that there are one, two, or three distinct positions of equilibrium, according as the ratio of the density of the tetrahedron to that of the water is greater, equal to, or less than 4 : 27.

24. A hemispherical shell (radius $2a$) containing water rotates with an angular velocity $\sqrt{\frac{3g}{7a}}$ about its axis which is vertical: a sphere (radius a) rests on the water with its lowest point in contact with the shell without pressure on it. If the free surface passes through the rim of the shell, shew that

density of sphere : density of water :: 128 : 189.

25. An isosceles triangular lamina ABC, right-angled at C, floats with its plane vertical and the angle C immersed, in a liquid of which the density varies as the depth; prove that, if $\frac{\pi}{4} + \theta$ be the angle which AB makes with the vertical, in either of the positions of equilibrium in which AB is not horizontal, the value of θ is given by an equation of the form

$$m \sin^2\theta \cos^2\theta = (\sin\theta + \cos\theta)^2.$$

26. A right circular cylinder, whose axis is vertical, contains a quantity of liquid, the density of which varies as the depth, and a right cone whose axis is coincident with that of the cylinder and which is of equal base, is allowed to sink slowly into the liquid with its vertex downwards. If the cone be in equilibrium when just immersed, prove that the density of the cone is equal to the initial density of the liquid at a depth equal to $\frac{1}{12}$th the length of the axis of the cone.

27. A quantity of liquid, the density of which varies as the depth, fills an inverted paraboloid, of latus rectum c, to a height h; prove that, if it be poured into a vessel of the form generated by the revolution round the axis of x of the curve,

$$a^4y^2 = 2ch^2x(a-x)(2a-x),$$

where a is any constant, its density will vary as the square of its depth.

28. A solid cone, of height h, vertical angle 2α, and density ρ, is moveable about its vertex, and its vertex is fixed at a depth c below the surface of a liquid, the density of which, at a depth z, is μz. The cone is in equilibrium with its axis inclined at an angle θ to the vertical, and its base above the surface; prove that

$$\mu c^5 \cos^8\alpha \cos\theta = 5\rho h^4 (\cos\overline{\theta+\alpha}\cos\overline{\theta-\alpha})^{\frac{5}{2}}.$$

CHAPTER V.

The stability of the equilibrium of floating bodies.

57. If a floating body be slightly displaced, it will in general either tend to return to its original position, or will recede farther from that position; in the former case the equilibrium is said to be *stable*, and in the latter *unstable*, for that particular direction of displacement.

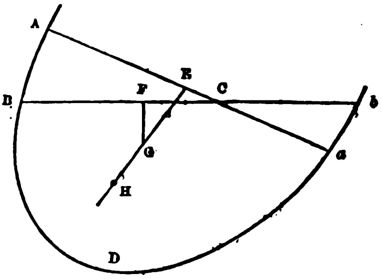
Consider first a small vertical displacement: it is clear that, if the body be floating partially immersed in homogeneous fluid, or if it be immersed, either wholly or partially, in a heterogeneous fluid of which the density increases with the depth, a depression will increase the weight of the fluid displaced, and on the contrary an elevation will diminish it; in either case the tendency of the fluid pressure is to restore the body to its position of rest, and the equilibrium is *stable* with regard to vertical displacements. This, it will be observed, is only shewn to be true of *rigid* bodies; if the increased pressure, caused by depression, have the effect of compressing any portion of the floating body, the equilibrium is not necessarily *stable*, and in fact it may be *unstable*.

An arbitrary displacement will in general involve both vertical and angular changes in the position of the body; if however the displacement be small, as we have supposed to be the case, the effects of the two changes of position can be treated independently; and we proceed to consider the effect of a small angular displacement, on the supposition that the weight of fluid displaced remains unchanged, and consequently that the fluid pressure has no tendency to raise or depress the centre of gravity of the body.

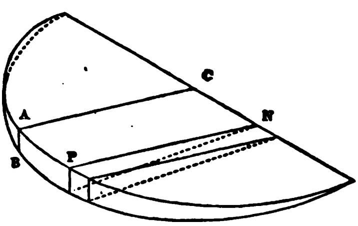
58. *A solid, floating at rest in a homogeneous liquid, is made to turn through a very small angle in a given vertical plane; to determine whether the fluid pressure will tend to restore it to its original position or not.*

Suppose the volume of liquid displaced to remain unchanged, and that the centre of buoyancy remains in the vertical plane of displacement through HG. This will be the case if CN be a principal axis of the plane of floatation.

Let AEC be the original plane of floatation and BCb the water-line after displacement through a small angle θ, G the centre of gravity of the solid, H of the fluid originally displaced, and V the volume of the fluid displaced.



In the second figure CN is the line of intersection of the two planes ACa, BCb, which is perpendicular to the plane ACB, in the first figure.



The resultant fluid pressure is the weight of $BDab$ acting upwards, and is therefore equivalent to the weight of ABa, or $g\rho V$, acting upwards through H, of the wedge aCb acting upwards, and of the wedge ACB acting downwards.

These wedges being equal, the resultant action of the two wedges is a couple, the moment of which about G is equal to its moment about C.

Taking for convenience $g\rho$ as unity, the weight of an element PN of one of the wedges

$$=\tfrac{1}{2}y^2\,\theta\delta x,$$

where $x = CN$, and $y = PN$; and the distance from CN of its centre of gravity is $\frac{2}{3}y$;

$$\therefore \text{ the moment about } CN \text{ of the wedges}$$

$$= \Sigma\left(\frac{1}{3}y^3\,\theta\,\delta x\right)$$

$$= \theta\int\frac{1}{3}y^3\,dx = \theta\,.\,Ak^2,$$

where A is the area of the section ACa of the body by the plane of floatation, and k its radius of gyration relative to the line CN. Hence the restorative moment of the fluid pressure about a horizontal axis through G, parallel to CN,

$$= (Ak^2 - V\,.\,HG)\,\theta;$$

and if this moment is positive the solid tends to return to its original position, i.e. the equilibrium is stable

$$\text{when } HG < \frac{Ak^2}{V},$$

and conversely, is unstable

$$\text{when } HG > \frac{Ak^2}{V}.$$

If M be the point in HG through which the resultant vertical pressure of the fluid acts, in other words, if the vertical line through the centre of buoyancy meet HG in M, the moment is

$$V\,.\,GM\sin\theta,$$

$$\text{or } V(HG - HM)\,\theta;$$

$$\therefore HM = \frac{Ak^2}{V},$$

and the equilibrium is stable or unstable according as $HM >$ or $< HG$.

The point M is called the *metacentre*.

If $HG = \dfrac{Ak^2}{V}$, that is, if M and G coincide, the equilibrium is said to be neutral.

Replacing $g\rho$, it will be seen that the restorative moment, for a displacement through a small angle θ, is

$$g\rho\theta\,(Ak^2 - V.HG).$$

59. We have assumed, in the preceding investigation, that the centre of gravity of the displaced liquid remains in the vertical plane of displacement passing through HG; when this is not the case, the expression

$$g\rho\,(Ak^2 - V.HG)\,\theta,$$

will still represent the moment of the fluid pressures, but the line of action of the resultant fluid pressure will not necessarily lie in the plane ABa.

Let $\bar{x}$ be the distance measured in the direction CN, 2nd figure, of the vertical through the centre of gravity (H') of the solid Bab, then

$$V\bar{x} = \int g\rho\theta\, xy\, dA,$$

so that $\bar{x}$ depends upon the product of inertia of the area, and vanishes when Cx and Cy are principal axes.

If the projection of the vertical through H' on the plane ABa meet HG in M, the moment of the fluid pressures about G will still be represented by $V.GM\theta$, and therefore as in the previous case $V.HM = k^2A$, and if rotation in the direction of the plane ABa only be allowed, the position of the point M defines the stability of the equilibrium.

60. It must be observed that the above investigation is essentially statical; it is simply an inquiry into the direction in which the moment of the fluid pressure about a certain horizontal axis through G is acting in the position of displacement contemplated.

Considered dynamically, if the horizontal axis through G be not a principal axis, the forces introduced by displacement will cause accelerations about other axes through G, and will consequently produce rotations about varying axes.

Moreover a rotation about G would, except in the case in which E and C, Art. 58, are coincident, cause a change in the quantity of fluid displaced, and vertical oscillations would therefore ensue.

61. The question of stability may be treated somewhat differently.

Defining the *metacentre* as the point of intersection with the line HG of the vertical line through the new centre of buoyancy after a slight displacement, we are led by help of Art. (54) to the following theorem;

The metacentre is the centre of curvature of the curve of buoyancy at the point in the same vertical line with G.

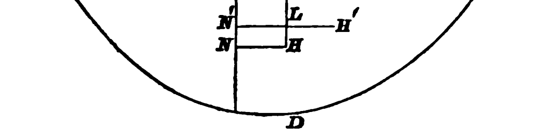
This is at once obvious from the fact that the point M is the point of intersection of consecutive normals to the curve.

Hence it appears that for any displacement, consistent with the condition that the volume displaced remains the same, the direction of the fluid pressure is always a vertical tangent to the evolute of the curve of buoyancy.

62. From the preceding theorem we can determine the expression for the height of the metacentre above the point H.

Let H be the centre of gravity of the volume ADB, and H' of aDb, aCA being a small angle θ.

Then, if α be an element of the area of the plane of floatation,



$H'N'$, HN, perpendiculars upon the vertical line through C,

$$H'N'.V - HN.V = \Sigma(Cn.\theta.\alpha.Cn) + \Sigma(Cn'.\theta.\alpha'.Cn'),$$

$$\text{or } H'L.V = \theta A k^2;$$

but if M be the centre of curvature at H,

$$H'L = H'M.\theta = HM.\theta,$$

$$\therefore\ V.HM = k^2A.$$

The restorative moment, for a small displacement θ,

$$= g\rho V.HM.\theta = g\rho\theta\,(Ak^2 - V.HG).$$

63. The preceding article assumes that the vertical line of action of the fluid pressure, after a slight displacement, intersects HG. This will be true only when the plane of displacement is a principal section, at H, of the surface of buoyancy. When this is not the case, the projection of the line of action on the vertical plane of displacement will intersect HG in a point M, which will be the centre of curvature of the normal section of the surface.

The radius of curvature of any normal section at H, of the surface of buoyancy, is therefore $\frac{Ak^2}{V}$, and, if I and I' be the principal moments of inertia of the plane of floatation at its centre of gravity, the principal radii of curvature, at H, of the surface of buoyancy are

$$\frac{I}{V} \text{ and } \frac{I'}{V},$$

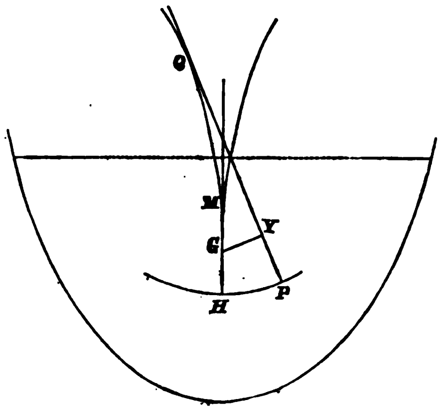
and the principal sections are parallel to the principal axes of the plane of floatation.

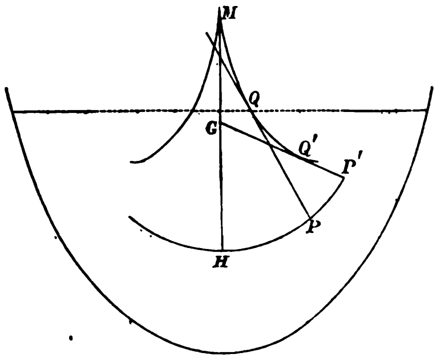
64. A most important case naturally presents itself; that is, the question of the stability of equilibrium of a ship when displaced by rolling.

In this case the vertical plane through HG, perpendicular to the plane of displacement, divides the floating body symmetrically, and consequently the vertical line HG passes through the point C in the plane of floatation.

The line HG also divides the curve of buoyancy symmetrically, and the point H is a point of maximum or minimum

curvature. In the first of these two cases the cusp of the evolute is pointed downwards; in the second case it is pointed upwards.





The figures at once shew the effects of displacement.

In the first case the *righting moment*, which is the *statical measure of stability* for a given angle of displacement, is proportional to GY the perpendicular from G on the tangent PQ, and increases with an increase in the angle of displacement.

In the second case, the righting moment increases to a maximum value, and then diminishes, vanishing for the position given by the tangent $GQ'P'$.

This is a position of equilibrium, but it is of unstable equi-

librium, in accordance with the general mechanical law that positions of stable and unstable equilibrium occur alternately.

If the equation to the curve of buoyancy be obtained in the form $p = f(\phi)$, G being the origin,

$$GY = \frac{dp}{d\phi},$$

and the righting moment is

$$W\frac{dp}{d\phi},$$

if W be the weight of the ship.

It is of course possible that the curve of buoyancy might have its concavity downwards; this will be the case when the sides of the ship, near the water-line, bend inwards.

In general the curve of buoyancy, for moderate displacements, is approximately an arc of an hyperbola; in the case of a 'wall-sided' ship, that is of a ship with the sides vertical near the water-line, the curve is an arc of a parabola.

65. Def. The surface which is the envelope of the planes cutting off a constant volume is called the *surface of floatation.*

Taking the case of a ship floating upright, the expression for the radius of curvature of a transverse section of the surface of floatation is

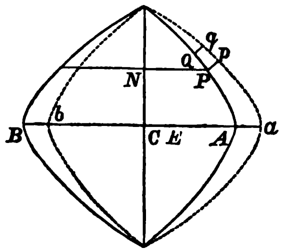
$$r_1 = \frac{\int y^2 \tan \alpha ds}{A},$$

ds being an element of the perimeter of the water-section, and α the inclination of the side of the ship to the vertical.

To prove this, let C' be the centre of gravity of a section through C making a small angle θ with the water-section ACB, and let aCb be the projection of the perimeter of the new section upon the water-section, E being the projection of C'.

Taking $PQ = ds$, and drawing Pp and Qq normals to the perimeter, the element of area $PQqp = y\theta \tan \alpha ds$;

$$\therefore\ CE(A) = 2\int y^2\theta \tan \alpha ds,$$



and, since $CC' = r_1\theta$, and $CE = CC'$ ultimately, it follows that

$$r_1 A = \int y^2 \tan a ds,$$

an expression first given by Mons. C. Dupin, in a memoir given to the Académie des Sciences in 1814.

A corresponding expression obviously exists for the radius of curvature (R_1) of the longitudinal section.

66. Calling r and R the metacentric heights for transversal and longitudinal displacements, that is, the radii of curvature of transverse and longitudinal sections of the surface of buoyancy; we know that

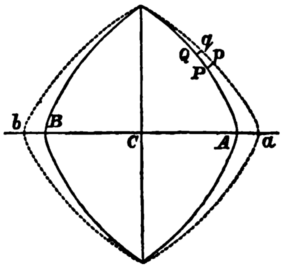
$$r = \frac{i}{V} \text{ and } R = \frac{I}{V},$$

where i and I are the principal moments of inertia of the water-section.

Mons. E. Leclert has established the following relations between these quantities;

$$r_1 = \frac{di}{dV} = r + \frac{Vdr}{dV}; \quad R_1 = \frac{dI}{dV} = R + V\frac{dR}{dV}.$$

A translation of Leclert's paper is given by Mr Merrifield in the *Proceedings*, for 1870, *of the Institution of Naval Archi-*



tects, and in the *Messenger of Mathematics*, March, 1872. The following is the first of the two proofs which are given.

Taking a section parallel to the water-section, and at a distance dz from it,

$$dV = A dz.$$

Let $apqb$ be the projection of this new section upon the water-section; then di is the moment of inertia of the area between ab and AB;

$$\therefore\ di = \Sigma y^2 dz \,.\, \tan\alpha\, ds,$$

$$\text{and } \frac{di}{dz} = \int y^2 \tan\alpha\, ds.$$

Hence
$$r_1 = \frac{1}{A}\frac{di}{dz} = \frac{di}{dV};$$

$$\therefore\ r_1 - r = \frac{di}{dV} - \frac{i}{V} = \frac{Vdi - idV}{VdV},$$

$$\text{or } r_1 = r + \frac{Vdr}{dV}.$$

67. We now append some examples of the determination of the metacentre.

Ex. 1. *A solid cylinder of radius a and length h floating with its axis vertical.*

In this case the plane of floatation is a circular area, and

$$Ak^2 = 4\int_0^a \frac{1}{3} y^3 dx = \frac{4}{3}\int_0^a (a^2 - x^2)^{\frac{3}{2}} dx$$

$$= \frac{4}{3} a^4 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta, \text{ putting } x = a\sin\theta,$$

$$= \frac{\pi a^4}{4};$$

therefore, if h' be the length of the axis immersed,

$$\pi a^2 h' \,.\, HM = \frac{\pi a^4}{4}, \text{ or } HM = \frac{a^2}{4h'},$$

and the equilibrium is stable if

$$\frac{a^2}{4h'} > \frac{h}{2} - \frac{h'}{2}.$$

Ex. 2. *A cylinder floating with its axis horizontal and in the surface is displaced in the vertical plane through the axis.*

The plane of floatation is a rectangle, and

$$Ak^2 = \frac{1}{6} ah^3,$$

h being the length of the cylinder, and a its radius;

$$\therefore\ HM = \frac{1}{3}\frac{h^2}{\pi a};$$

and the equilibrium is stable, if

$$\frac{1}{3}\frac{h^2}{\pi a} > \frac{4a}{3\pi},$$

$$\text{or } h > 2a.$$

Ex. 3. *A solid cone floating with its axis vertical and vertex downwards.*

Let h be the length of the axis,

z the portion of the axis immersed,

2α the vertical angle of the cone.

Then
$$Ak^2 = \frac{1}{4}\pi z^4 \tan^4 \alpha,$$

$$\text{and } V = \frac{1}{3}\pi z^3 \tan^2 \alpha;$$

$$\therefore\ HM = \frac{3}{4} z \tan^2 \alpha;$$

$$\text{also } HG = \frac{3}{4}h - \frac{3}{4}z,$$

and therefore the equilibrium is stable or unstable, according as

$$z \tan^2 \alpha > \text{or} < h - z,$$

$$\text{or } \quad z > \text{or} < h \cos^2 \alpha.$$

But if ρ, σ, be the densities of the fluid and cone,

$$\left(\frac{z}{h}\right)^3 = \frac{\sigma}{\rho};$$

therefore the equilibrium is stable or unstable as

$$\frac{\sigma}{\rho} > \text{or} < (\cos \alpha)^6.$$

Ex. 4. *An isosceles triangular prism floating with its base not immersed, and its edges horizontal.*

Referring to Art. 48, consider first the position of equilibrium in which the base is inclined to the horizon.

In this case, if $AQ = 2y$ and $AP = 2x$, x and y are given by the equations

$$x + y = 2a\cos^2\frac{\theta}{2},$$

$$xy = c^2.$$

The co-ordinates of G and H referred to AB, AC as axes are respectively,

$$\frac{2}{3}a,\quad \frac{2}{3}a,\quad \text{and}\ \frac{2}{3}x,\quad \frac{2}{3}y,$$

$$\therefore\ HG^2 = \frac{4}{9}\{(a-x)^2 + (a-y)^2 + 2(a-x)(a-y)\cos\theta\}$$

$$= \frac{4}{9}\{x^2 + y^2 + 2xy\cos\theta - 2a(1+\cos\theta)(x+y) + 2a^2(1+\cos\theta)\},$$

from which, by means of the above equations, we obtain

$$HG = \frac{4}{3}\sin\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right)^{\frac{1}{2}}.$$

The area $PAQ = 2c^2\sin\theta$, and if M be the metacentre, and l the length of the prism,

$$2lc^2\sin\theta\,.\,HM = \frac{PQ^2}{12}\,.\,PQ\,.\,l,$$

$$HM = \frac{PQ^3}{24c^2\sin\theta}.$$

But $$PQ^2 = 4(x^2 + y^2 - 2xy\cos\theta)$$

$$= 16\cos^2\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right);$$

$$\therefore\ HM = \frac{4}{3}\,\frac{\cos^2\frac{\theta}{2}}{c^2\sin\frac{\theta}{2}}\left(a^2\cos^2\frac{\theta}{2} - c^2\right)^{\frac{3}{2}},$$

$$\text{and } HM > HG, \text{ if } c^2 \sin^2\frac{\theta}{2} < \cos^2\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right),$$

$$\text{i.e. if } \cos^2\frac{\theta}{2} > \frac{c}{a}.$$

Next, consider the case in which the base is horizontal, and PQ therefore parallel to BC.

The area $PAQ = 2c^2 \sin\theta$,

$$AP = AQ = 2c, \quad \text{and } PQ = 4c \sin\frac{\theta}{2}.$$

$$\text{Hence, } HM = \frac{4}{3}c\frac{\sin^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}, \text{ and } HG = \frac{4}{3}(a-c)\cos\frac{\theta}{2},$$

$$\text{and } HM > HG \text{ if } \cos^2\frac{\theta}{2} < \frac{c}{a}.$$

Now in the Art. 48, before referred to, we have shewn that there are three positions of equilibrium, or one only, according as

$$\cos^2\frac{\theta}{2} > \text{ or } < \frac{c}{a}.$$

Hence it follows, that when there are three positions of equilibrium, the intermediate one, in which CB is horizontal, is a position of unstable equilibrium, while in the other two positions the equilibrium is stable.

If there be only one position in which the prism will rest, its equilibrium is stable.

It will be a useful exercise for the student to obtain these results by investigating the equation to the curve of buoyancy, and determining the position of its centre of curvature.

68. *Finite displacements.* If a solid body, floating in water, be turned through any given angle from its position of equilibrium, then, as before, the moment of the fluid pressure is restorative or not according as the point L at which the vertical through the new centre of buoyancy meets the line HG is above or below G.

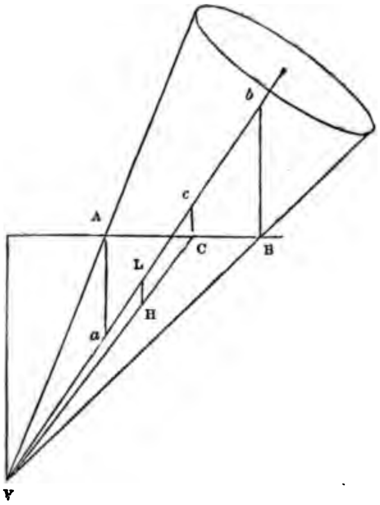
It is not to be inferred that if L is above G, the body will when set free return to its original position and oscillate through it, or even that the original position is one of stable equilibrium, according to our previous definition of stability: it is a general law of mechanics that positions of stable and unstable equilibrium occur alternately, and the body may have been displaced from its original position *through* other positions of equilibrium.

As a particular example take the following.

A solid cone, floating with its axis vertical and vertex downwards, is turned through an angle θ in a vertical plane, the volume of fluid displaced remaining the same; to determine the direction of the moment of the fluid pressure.

Let AB be the major axis of the elliptic section made by the surface plane of the fluid, C its middle point, Aa, Bb, Cc, lines at right angles to AB, and let the angle $AVB = 2\alpha$ and $VA = d$. Then

$$VAa = \theta - \alpha; \text{ and } VBb = \pi - \theta - \alpha.$$



$$Vc = \frac{1}{2}(Va + Vb) = \frac{1}{2}\cdot\left\{d\frac{\sin(\theta-\alpha)}{\sin\theta} + d\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\frac{\sin(\theta+\alpha)}{\sin\theta}\right\}$$

$$= \frac{d\cos\theta}{\cos(\theta+\alpha)};$$

$$\therefore\ VL = \frac{3}{4}d\,\frac{\cos\theta}{\cos(\theta+\alpha)}.$$

The semi-minor axis of the ellipse AB is a mean proportional between the perpendiculars from A and B on the axis of the cone,

$$\therefore\ \text{its area} = \pi\frac{1}{2}AB\,(VA\,.\,VB\,.\sin^2\alpha)^{\frac{1}{2}}$$

$$= \frac{\pi}{2}d^2\,\frac{\sin\alpha\sin 2\alpha}{\cos(\theta+\alpha)}\cdot\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{1}{2}};$$

therefore the volume of the fluid displaced

$$= \frac{1}{3}d\cos(\theta-\alpha)\,.\,(\text{area of ellipse})$$

$$= \frac{1}{3}\pi d^3\sin^2\alpha\cos\alpha\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{3}{2}}.$$

Hence, if ρ, σ, be the densities of the fluid and the cone, since the weight of the fluid displaced is equal to that of the cone, we have

$$\rho d^3\sin^2\alpha\cos\alpha\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{3}{2}} = \sigma h^3\tan^2\alpha,$$

$$\text{or}\ \left(\frac{d}{h}\right)^3 = \frac{\sigma}{\rho}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}\frac{1}{\cos^3\alpha}.$$

And $VL > VG$ if

$$d\,\frac{\cos\theta}{\cos(\theta+\alpha)} > h,$$

or if

$$\sqrt[3]{\frac{\sigma}{\rho}} > \frac{\cos\alpha\cos(\theta+\alpha)}{\cos\theta}\cdot\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{1}{2}}.$$

Supposing θ indefinitely small, we obtain the condition of stability for an infinitesimal displacement,

$$\sqrt[3]{\frac{\sigma}{\rho}} > \cos^2\alpha;\ \text{as before, Ex. 3, Art. 68.}$$

Let the equilibrium of the cone be neutral, that is, let

$$\sigma = \rho \cos^6 \alpha,$$

then, after a finite displacement, the action of the fluid will tend to restore the cone to its original position, if

$$\cos \alpha \,.\, \cos \theta > \sqrt{\{\cos (\theta + \alpha) \,.\, \cos (\theta - \alpha)\}},$$

a condition which is always true, α and θ being each less than a right angle.

In the case of neutral equilibrium of a cone, the equilibrium may therefore be characterised as stable for any finite displacement.

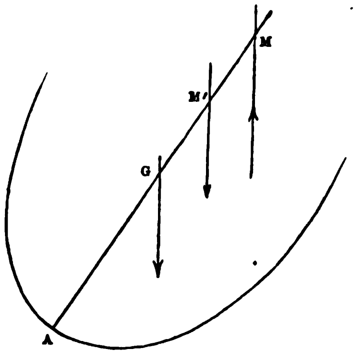
69. When liquid is contained in a vessel, which is slightly displaced from its original position, the preceding investigations enable us to determine the line of action of the resultant *downward* pressure.

The problem in fact in this case, as in the previous case, is the following.

A given volume, the centre of gravity of which is H, is cut from a solid ABC by a plane, and the line CH is perpendicular to the plane; the same volume being cut off by a plane making a very small angle with the plane AB, to determine the position of the straight line perpendicular to the second plane, and passing through the centre of gravity of the volume cut off by it.

If the interior surface of the vessel is symmetrical with respect to the plane through H perpendicular to the line of intersection of the two planes, the line whose position is required will intersect CH in a point M, *the metacentre*, the position of which is determined by our previous results.

70. *A hollow vessel containing liquid, floats in liquid; required to determine the nature of the equilibrium, supposing that the body is symmetrical with respect to the vertical plane of displacement through its centre of gravity, and that the centres of gravity of the body and of the liquid are in the same vertical line.*



Let M be the metacentre for the displaced fluid, and M' for the contained fluid, W, W', the weights of the displaced and contained fluid*.

Taking moments about G, the centre of gravity of the vessel, the resultant fluid pressures will tend to restore equilibrium, or the reverse, according as

$$W \,.\, GM - W' \,.\, GM'$$

is positive or negative, i.e. as

$$\frac{W}{W'} > \text{ or } < \frac{GM'}{GM}.$$

Ex. *A hollow cone containing water floats in water with its axis vertical.*

Let h = the length of the axis of the cone,

h' = the length of the axis in the contained fluid,

z = the length beneath the surface of the external fluid.

Taking 2α as the vertical angle of the cone, we have

$$HM = \frac{3}{4} z \tan^2\alpha, \text{ Art. 59, Ex. 3.}$$

But $$HG = \frac{2}{3} h - \frac{3}{4} z;$$

* This is the case of a leaky ship rolling; the next article discusses the pitching of a leaky ship.

$$\therefore\ GM = \frac{3}{4} z \sec^2 \alpha - \frac{2}{3} h.$$

Similarly $$GM' = \frac{3}{4} h' \sec^2 \alpha - \frac{2}{3} h,$$

also $$\frac{W}{W'} = \frac{z^3}{h'^3};$$

therefore the equilibrium is stable if

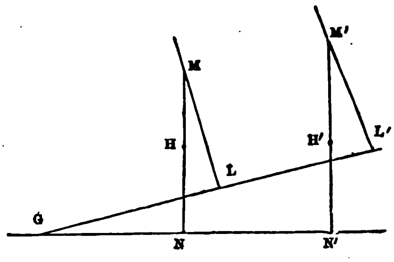
$$\left(\frac{z}{h'}\right)^3 > \frac{9h' \sec^2 \alpha - 8h}{9z \sec^2 \alpha - 8h},$$

z being given by the equation

$$W - W' = \frac{1}{3} g\rho\pi \tan^2 \alpha \,(z^3 - h'^3) = \text{weight of cone.}$$

71. In the case in which the centres of gravity of the contained and of the fluid displaced are not in the same vertical, suppose the displacement to take place in direction of the vertical plane through the centres of gravity, and that the body is symmetrical with respect to that plane.

Let G be the centre of gravity of the body, H of the fluid displaced, H' of the contained fluid, and M, M', the metacentres.



Also let GNN' be horizontal in the position of equilibrium, and GLL' the horizontal line through G in the displaced position.

Then W, W', having the same meanings as before, and θ being the angle of displacement, the equilibrium is stable or unstable, as

$$W \,.\, GL > \text{or} < W' \,.\, GL',$$

$$\text{or } W(GN\cos\theta + MN\sin\theta) > \text{or} < W'(GN'\cos\theta + M'N'\sin\theta),$$

$$\text{i.e. since } W \,.\, GN = W' \,.\, GN',$$

as
$$\frac{W}{W'} > \text{or} < \frac{M'N'}{MN}.$$

72. *Stability of the equilibrium of bodies floating under constraint.*

In those cases of constraint, in which, for a small displacement, the volume of liquid displaced remains unchanged, the theory of the metacentre determines the line of action of the fluid pressure, and the question of stability is then easily determined.

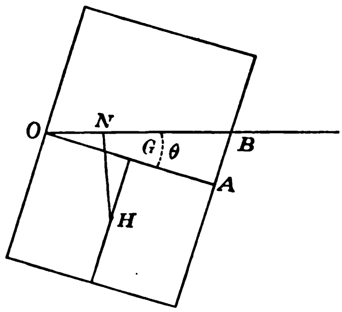
Suppose, for instance, that a body, partially immersed, is moveable about a horizontal axis, which is vertically beneath the centre of gravity (C) of the plane of section of the body by the surface of the liquid.

The effect of a displacement through a small angle θ will be to depress the point C through a space which depends upon θ^2, and therefore, to the first order of small quantities, the volume displaced remains unchanged, and the metacentre is the same as if C remained in the surface.

If the body be moveable about a horizontal axis which is not vertically beneath the point C, the change in the volume displaced cannot be neglected, and the question of stability must be treated by a direct consideration of the action of the displaced liquid.

Ex. *A rectangular lamina rests in a liquid of twice its own density with two of its sides vertical, and is moveable in its own plane about the middle point of one of its vertical sides.*

The figure represents the lamina when slightly displaced through an angle AOB, (θ), the point O which is in the surface being the middle point.



Then if $OA = a$, and if the height $= 2b$, the area

$$AOB = \frac{1}{2} a^2 \theta,$$

and, taking moments about O, the equilibrium is stable if

$$2\rho \left(\frac{1}{2} a^2 \theta \cdot \frac{2}{3} a + ab \cdot ON\right) > \rho \cdot 2ab \cdot \frac{a}{2},$$

HN being the vertical through H;

$$\text{or, since } ON = OG \cos\theta - HG \sin\theta = \frac{a}{2} - \frac{b}{2}\theta,$$

$$\text{if, } 2a^2 > 3b^2.$$

73. *The equilibrium of bodies floating in two liquids.*

Suppose the body to be wholly immersed with the portion V of its volume in the upper liquid and V' in the lower.

Take the case in which the centres of gravity H, H', of the liquids displaced by V and V', and therefore the centre of gravity G of the body, are in the same vertical.

Displace the body through a small angle θ about an axis through C the centre of gravity of the section ACB of the body by the common surface of the liquids.

The action then consists of a pressure $g\rho' V'$ acting upwards through M' the metacentre for the lower liquid, and of a pressure $g\rho V$ acting upwards through M the metacentre for the upper liquid.

If M be above G and M' below G, the equilibrium is stable, if

$$g\rho V . GM > g\rho' V' . GM',$$

or, observing that HM is measured downwards from H,

$$\text{if}\quad \rho V . HG - \rho Ak^2 > \rho' V' H' G - \rho' Ak^2,$$

$$\text{or}\quad (\rho' - \rho) Ak^2 > \rho' V' . H' G - \rho V . HG,$$

Ak^2 being the moment of inertia of the area ACB about the axis through C.

The treatment of the problem is the same for other relative positions of M, G and M'.

When the points H, G, H' are not in the same vertical, the metacentres lie in the perpendiculars from H and H' on ACB, and the nature of the equilibrium is determined as above by comparing the moments about G.

74. The preceding question may be also usefully treated in the following manner.

The body may be supposed to be completely immersed in a liquid of density ρ, and we can then imagine a liquid of density $\rho' - \rho$ superposed.

Let E be the centre of gravity of the whole volume $V + V'$, then

$$(V + V') . EG = V' . H' G - V . HG.$$

If the body be displaced through a small angle θ, the restorative moment is

$$g(\rho' - \rho)(Ak^2 - V' . H' G) - g\rho (V + V') . EG,$$

which by the above relation becomes

$$g(\rho' - \rho) Ak^2 - g\rho' V' . H' G + g\rho V . HG,$$

and the stability depends upon the sign of this expression.

75. *Stability of a body floating in heterogeneous liquid.*

We shall consider only the case in which the body is symmetrical with regard to the line HG so that this line passes through C, the centre of gravity of the water-section, and

contains the centre of gravity of all the strata of liquid displaced.

The effect of this limitation is that a small displacement (θ) about C, or about any point in HG, raises the centre of gravity of any horizontal section through a height of the order θ^2, and we can therefore employ the formulæ of Arts. (58) and (73).

76. *Determination of the metacentre for a body floating in heterogeneous liquid.*

A liquid in which the density is a function of the depth, can be conceived as made up of a series of homogeneous liquids having successive descending surfaces, and the centre of gravity H of the whole mass U displaced, will be the centre of gravity of the aggregate of these liquids.

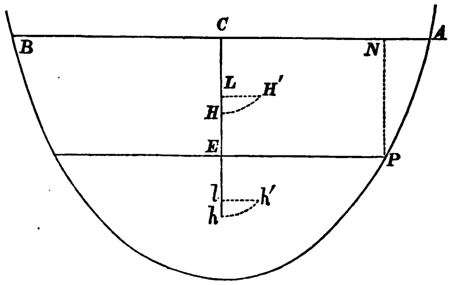
Turning the body through a small angle θ, the vertical tilt of the centre of gravity of each portion will vary as θ^2, and therefore the vertical tilt of H will vary as θ^2.

If E be the surface, ρ the density, and V the volume displaced of one of these liquids, and if h be the centre of gravity of V, then, as in Art. (62),

$$h'l \, . \, V = \theta A k^2,$$

A being the area of the section E;

therefore $$H'L \, . \, U = \Sigma h'l \, . \, V\rho = \theta \Sigma \rho A k^2.$$



Further, if M be the metacentre,

$$H'L = H'M.\theta = HM.\theta,$$

and therefore $$HM.U = \Sigma\rho Ak^2.$$

This formula includes the case in which the solid bulges out below the water-section.

Taking CA as the axis of y, and considering the case when the solid does not bulge out,

$$\Sigma\rho Ak^2 = \iint\Sigma(\rho)\, y^2\, dxdy,$$

the double integration extending over the water-section, and the summation of ρ down the vertical ordinate NP; hence, if ρ' be the density at P,

$$HM.U = \iint\rho' y^2\, dxdy.$$

77. If the floating body be a solid of revolution, having its axis vertical, the formulæ can be somewhat simplified.

For, transferring to polar co-ordinates,

$$HM.U = 4\int_0^{\frac{\pi}{2}}\int_0^a \rho' r^3 \sin^2\theta dr\, d\theta$$
$$= \int_0^a \pi\rho' r^3 dr,$$

ρ' being the density corresponding to the section of radius r.

Suppose, for example, that the density varies as the depth, and that the floating body is a cone, vertex downwards.

If h be the length of axis immersed,

$$U.HM = \int_0^{h\tan\alpha} \pi\mu\,(h - r\cos\alpha)\, r^3 dr = \frac{\pi\mu h^5 \tan^4\alpha}{20},$$

and $$U = \int_0^h \pi\mu z\,(h-z)^2 \tan^2\alpha dz = \frac{\pi\mu h^4 \tan^2\alpha}{12},$$

$$\therefore\ HM = \frac{3}{5} h \tan^2\alpha.$$

Also, if V be the vertex,

$$VH = \frac{3}{5} h;$$

$$\therefore\ VM = \frac{3}{5} h \sec^2\alpha.$$

EXAMPLES.

1. An inverted vessel formed of a substance which is heavier than water contains enough air to make it float; prove that, if it be pushed down through a certain space, it will be in a position of equilibrium which for vertical displacement will be unstable.

2. A solid cylinder, one end of which is rounded off in the form of a hemisphere, floats with the spherical surface partly immersed: find the greatest height of the cylinder which is consistent with stability of equilibrium.

3. A cone, whose vertical angle is $60°$, floats in water with its axis vertical and vertex downwards; shew that its metacentre lies in the plane of floatation; and that its equilibrium will be stable provided its specific gravity $> \frac{27}{64}$.

4. An isosceles wedge floats with its base horizontal, and its edge immersed; shew that the equilibrium is stable for displacements in a plane perpendicular to the edge, if the ratio of the density of the wedge to that of the fluid is greater than the ratio $(\cos a)^4 : 1$, $2a$ being the angle of the wedge.

5. A closed cylindrical vessel, quarter-filled with ice, is placed floating in water with its axis vertical; the weight of the vessel is one-fourth of the weight of the water which it can contain; examine the nature of the equilibrium before and after the ice melts, neglecting the change of volume consequent on the change of temperature.

6. Find a solid of revolution such that, when a segment of it is immersed in liquid, the distance between the centre of buoyancy and the metacentre may be constant, whatever be the height of the segment.

7. Water rests upon mercury, and a cone is too heavy to rest without its vertex penetrating the mercury; find the density of the cone that the equilibrium may be stable.

8. If a cylindrical shell without weight contain liquid and float in another liquid, shew that the equilibrium will be stable, unless the ratio of the density of the internal to the external fluid is less than unity, and greater than half the duplicate ratio of the radius of the cylinder to the depth of the internal fluid.

9. A hemispherical shell, containing liquid, is placed on the vertex of a fixed rough sphere of twice its diameter; prove that the equilibrium will be stable or unstable, as the weight of the shell is greater or less than twice the weight of the liquid.

10. A solid of revolution floats with its vertex downwards, determine its form when the position of the metacentre is independent of the density of the liquid.

11. A conical shell, vertex downwards, floats in unstable equilibrium; how much water must be poured in to make the equilibrium stable?

12. A cone is placed in water with its axis vertical and its vertex resting on the horizontal base of the vessel containing the water; find the least depth of water consistent with stable equilibrium.

13. A cylindrical vessel, the weight of which may be neglected, contains water, and the vessel is placed on the vertex of a fixed rough sphere with the centre of its base in contact with the sphere. Find the condition of stability for infinitesimal displacements, and prove that, if the equilibrium be neutral for such displacements, it will be unstable for small finite displacements.

14. Find the form of a solid of revolution floating with its axis vertical, and such that the distances of the metacentre and the centre of buoyancy from the lowest end of the solid may be in a constant ratio whatever be the density of the liquid.

15. A right circular cone floats with its axis horizontal in a liquid the density of which is double that of the cone, the vertex being attached to a fixed point in the surface of the liquid; prove that for stability the vertical angle must be less than 120°.

16. A cylindrical vessel is moveable about a horizontal axis passing through its centre of gravity, and is placed so as to have its axis vertical; if water be poured in, shew that the equilibrium is

at first unstable; and find the condition which must be satisfied, in order that it may be possible to make the equilibrium stable by pouring in enough water.

17. A thin conical vessel of given weight is moveable about a diameter of its base, which is horizontal, and is partly filled with a heavy fluid; shew that the equilibrium is always stable if the semi-vertical angle of the cone is $< \cdot 30^\circ$; and if it be greater than this, determine when the equilibrium is stable or unstable.

18. Water is contained in a vessel having a horizontal base, and a paraboloid whose specific gravity is four-ninths that of water, and the length of whose axis is to the latus rectum as nine to eight, is supported partly by the fluid and partly by the base on which the vertex rests; find the least depth of the fluid for which the equilibrium is stable.

19. A parabolical cup, the weight of which is W, standing on a horizontal table, contains a quantity of water, the weight of which is nW; if h be the height of the centre of gravity of the cup and the contained water, the equilibrium will be stable provided the latus rectum of the parabola be

$$> 2(n+1)h.$$

20. A solid of revolution floats with its axis vertical, and is sunk to different depths by placing weights at a fixed point on its axis.

Find the form of the solid that the equilibrium may always be neutral.

21. A solid cone whose axis is vertical and vertex downwards is moveable about an axis coincident with a generating line; to what depth must the system be immersed in water, in order that the equilibrium of the cone may be stable?

22. A solid of revolution possesses this property. A portion being cut off by a plane perpendicular to its axis and immersed vertex downwards in a liquid and then displaced through a small angle, the moment tending to restore equilibrium is independent of the amount cut off. Shew that, if $y = f(x)$ be the generating curve, to determine f we have

$$[f(x)]^2 = \rho\,[1 + \{f'(x)\}^2 + f(x)f''(x)]\,[f\{x + f(x)f'(x)\}]^2,$$

being the density of the solid compared with the fluid.

23. The solid formed by a portion of $cy^2 = z(a^2 - x^2)$ cut off by a plane parallel to that of xy floats in a fluid of n times its density; prove that, if it is in neutral equilibrium for small angular displacements in any vertical plane,

$$n^{\frac{1}{2}} = 1 + \frac{5}{8}\frac{a^2}{c^2}.$$

24. An isosceles triangular lamina ABC floats with its base AB horizontal, and above the surface, in a liquid, the density of which varies as the depth: if h be the depth of C below the surface, the height of the metacentre above C is

$$\frac{1}{2} h \sec^2 \frac{C}{2}.$$

25. An elliptic lamina floats half immersed, with its transverse axis ($2a$) vertical, in a liquid, the density of which varies as the square of the depth; prove that the depth of the metacentre is

$$\frac{32}{15}\frac{ae^2}{\pi},$$

e being the eccentricity.

26. A right circular cylinder rests in a liquid with its axis vertical and a length c immersed. The density at a depth z being $\phi(z)$, shew that the depth of the metacentre is

$$\frac{\int_0^a z\phi(z)\,dz - \frac{a^2}{4}\phi(a)}{\int_0^a \phi(z)\,dz}.$$

27. A paraboloid of revolution floats with its axis vertical and vertex downwards in a liquid, the density of which varies as the depth; the equilibrium will be stable or unstable, according as $4c$ is less or greater than $3(m + a)$, where c is the length of the axis, a the length immersed, and m the latus rectum of the generating parabola.

28. A prolate spheroid floats half immersed, with its axis vertical, in a liquid, the density of which varies as the square of the depth; the height of the metacentre above the surface is

$$\frac{5}{8}\frac{a^2 - b^2}{b}.$$

29. A solid paraboloid of revolution floats with its axis vertical, vertex downwards, and focus in the surface of a liquid, the density of which at the depth z is $\mu(a+z)$, $4a$ being the latus rectum of the generating parabola; prove that the distance of the metacentre from the vertex is $\frac{21}{8}a$.

30. A homogeneous cone floats with its vertex downwards in a liquid whose density varies as the square of the depth; if the density of the cone be equal to that of the liquid at a depth equal to a fifth of the height of the cone, the vertical angle, when the equilibrium is neutral, is given by the equation,

$$\cos^2 \alpha = \frac{2}{3}\left(\frac{2}{5}\right)^{\frac{1}{5}}.$$

CHAPTER VI.

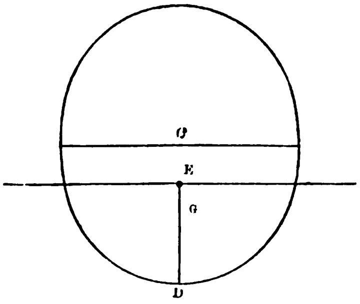
OSCILLATIONS OF FLOATING BODIES.

78. A HEAVY body which is floating in liquid in a position of stable equilibrium, will, if slightly displaced from that position, make small vertical and angular oscillations; we proceed to consider, in a simple case, the laws of these oscillations. We shall suppose that the body is symmetrical with regard to a vertical plane through its centre, and that the initial displacement is parallel to this plane.

It is evident that the subsequent motions of all points of the body will be parallel to this plane, and if the equilibrium be stable, that the motion will consist of small vertical and angular oscillations.

First, let the vertical line through G and H (CED) pass through the centre of gravity of the plane of floatation. When this is the case we can consider the vertical and angular displacements independently of each other.

Suppose a small vertical displacement; then the portion CE of the body which is raised out of the fluid may be considered as a thin cylinder.



Let $CE=z$, then $EG=CG-z$, and

the moving force downwards on the body = the weight of the body − the weight of the fluid displaced

$$=g\rho A\,.\,z,$$

if A be the area of the plane of floatation;

$$\therefore\ m\frac{d^2.EG}{dt^2}=g\rho Az,$$

m being the mass of the body.

But mg = the weight of fluid displaced

$=g\rho V$, V being the volume CD;

$$\therefore\ \frac{d^2z}{dt^2}+\frac{gA}{V}z=0,$$

is the equation which determines the motion.

The time of a complete oscillation is therefore $2\pi\sqrt{\left(\frac{V}{gA}\right)}$.

79. Next suppose a small angular displacement (α) about C, then G is raised through a space which depends on α^2, and therefore may be neglected in comparison with quantities depending upon α, and if the body, supposed at rest, be then left to itself, it will (on the supposition that the equilibrium is stable) oscillate about a horizontal axis through G.

It would in fact come to the same thing if the initial displacement were about G, as the point C would move sensibly (that is, considering small quantities of the first order only,) in a horizontal direction, and the quantity of fluid displaced would, as before, remain unchanged.

If M be the metacentre, the moment of the fluid pressure about G

$$=g\rho V.MG.\sin\theta,$$

and tends to diminish θ, the angle made by GH with the vertical at the time t.

But $$MG = \frac{k^2 A}{V} - a;\ \text{if } HG = a,$$

therefore, since the horizontal axis through G is a principal axis, we have

$$mK^2 \frac{d^2\theta}{dt^2} = -g\rho\,(k^2A - aV)\,\theta,$$

neglecting higher powers of θ, where mK^2 is the moment of inertia of the body about the horizontal axis through G,

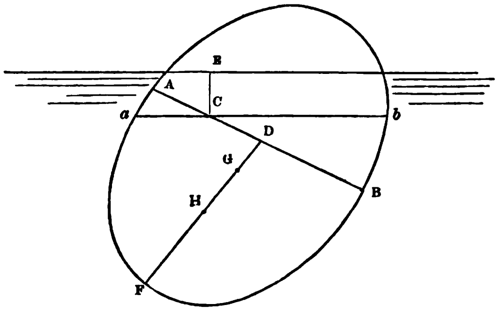
$$\text{or } K^2 \frac{d^2\theta}{dt^2} + g\left(\frac{k^2A}{V} - a\right)\theta = 0,$$

an equation which, when $k^2A > aV$, that is, when M is above G, indicates small oscillations taking place in the time

$$\pi K \sqrt{\left\{\frac{V}{g\,(k^2A - aV)}\right\}}.$$

If G is below H the sign of a will of course be changed.

80. Secondly, when the line joining H and G does not pass through C, the two motions are not independent, but the law which defines these motions can be determined as follows.



Suppose the body to be slightly displaced in the vertical plane of symmetry, and then left to itself; and at the time t let θ be the angle made by HG with the vertical, and $z = CE$ the depth of C below the surface.

Let HG meet the plane of floatation in D,

$$HG = a, \quad CD = b, \quad DG = c,$$

and other symbols as before.

Then the depth of $G = z + b\sin\theta + c\cos\theta$

$= z + b\theta + c$, to the order considered.

The weight of the fluid displaced is the weight of a volume of fluid equal to

$$aFb + EC, \text{ or } AFB + EC;$$

$$\text{this weight} = g\rho V + g\rho Az,$$

$$\text{and } \therefore\ m\frac{d^2}{dt^2}(z + c + b\theta) = mg - (g\rho V + g\rho Az)$$

$$= -g\rho Az;$$

$$\text{or } \frac{d^2z}{dt^2} + b\frac{d^2\theta}{dt^2} = -g\frac{A}{V}.z \text{ ....} \text{(I)}.$$

Another equation is to be obtained from the consideration of the angular motion about the horizontal axis through G, which is a principal axis, perpendicular to the plane of displacement.

The moment of the fluid pressure about G may be divided into two parts, the one due to the portion aFb, and the other to the portion EC of the fluid displaced.

The former part of the fluid pressure $= g\rho V$ acting upwards through M the metacentre; the latter $= g\rho Az$, and may be considered to act through C the centre of gravity of the plane of floatation.

The moment, in the direction tending to diminish θ,

$$= g\rho V.GM\sin\theta - g\rho Az(b\cos\theta - c\sin\theta)$$
$$= g\rho(k^2A - aV)\theta - g\rho Az(b - c\theta)$$
$$= g\rho(k^2A - aV)\theta - g\rho Abz,$$

neglecting the product of z and θ;

$$\therefore\ mK^2\frac{d^2\theta}{dt^2} = -g\rho(k^2A - aV)\theta + g\rho Abz.$$

$$K^2\frac{d^2\theta}{dt^2} = -g\left(\frac{k^2A}{V} - a\right)\theta + g\frac{A}{V}.bz \text{ ....} \text{(II)}.$$

From the equations (I) and (II) we obtain

$$\frac{d^2z}{dt^2}+\frac{gA}{V}\left(1+\frac{b^2}{K^2}\right)z-\frac{gb}{K^2}\left(\frac{k^2A}{V}-a\right)\theta=0,$$

$$\frac{d^2\theta}{dt^2}-\frac{gAb}{VK^2}z+\frac{g}{K^2}\left(\frac{k^2A}{V}-a\right)\theta=0,$$

which may be written

$$\frac{d^2z}{dt^2}+rz-bn\theta=0,$$

$$\frac{d^2\theta}{dt^2}-\frac{pz}{b}+n\theta=0.$$

To integrate these equations, multiply the second by λ, and add it to the first, then,

assuming $$\frac{\lambda n-bn}{rb-\lambda p}=\frac{\lambda}{b} \quad \text{(III)},$$

we have $$\frac{d^2}{dt^2}(z+\lambda\theta)+\left(r-\frac{\lambda p}{b}\right)(z+\lambda\theta)=0,$$

and if λ_1, λ_2 be the roots of (III)

$$z+\lambda_1\theta=C_1\cos\left\{\sqrt{r-\lambda_1\frac{p}{b}}\,t+\alpha_1\right\},$$

$$z+\lambda_2\theta=C_2\cos\left\{\sqrt{r-\lambda_2\frac{p}{b}}\,t+\alpha_2\right\},$$

from which z and θ are completely determined.

The depth of G is given by an expression of the form

$$C+A\cos(\mu t+\alpha)+B\cos(\mu' t+\beta),$$

and its motion consists of two distinct oscillations, each following the pendulum laws, and compounded together in accordance with the principle of the coexistence of small oscillations*.

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* Poisson's *Cours de Mécanique*, Art. 61P.

81. It may be observed that if two points be taken in the line AB, whose distances from C in the direction CD are λ_1, λ_2, then at the time t, the vertical depths of these points are $z+\lambda_1\theta$ and $z+\lambda_2\theta$, that is, are

$$C_1 \cos\left\{\sqrt{r-\lambda_1\frac{p}{b}}\,t+\alpha_1\right\}, \text{ and } C_2 \cos\left\{\sqrt{r-\lambda_2\frac{p}{b}}\,t+\alpha_2\right\},$$

and their vertical motions are therefore simple oscillations following the pendulum law. This remark is quoted by Duhamel (*Cours de Mécanique*, Art. 152) as due to M. Cauchy.

EXAMPLES.

1. A solid, the lower portion of whose surface is spherical, floats in a heavy fluid; shew that the time of a small angular oscillation is the same in whatever fluid it floats.

2. A hollow hemisphere moveable about a horizontal diameter is partly filled with fluid; shew that the time of a small oscillation is the same as if there were no fluid in it.

3. A solid ellipsoid floats in a liquid of twice its own specific gravity with its shortest axis vertical; find the time of a small vertical oscillation, and also the times of small angular oscillations about the two horizontal axes.

4. A cube (the length of whose edge is $2a$) is floating in a fluid with its centre of gravity at a depth c below the surface; if it receive a small displacement so that two of its faces remain vertical, shew that the times of its small vertical and angular oscillations are

$$\pi\sqrt{\left(\frac{a+c}{g}\right)} \text{ and } 2\pi\sqrt{\left\{\frac{a^2(a+c)}{g(3c^2+a^2)}\right\}}, \text{ respectively.}$$

5. A cone of vertical angle 2α floats in a cylinder of radius a with a length h of its axis immersed. If it be pushed vertically downwards through a small space, shew that the time of an oscillation is

$$\pi\sqrt{\frac{(a^2-h^2\tan^2\alpha)\,h}{3a^2g}}.$$

6. An oblate spheroid is completely immersed in two fluids, the specific gravity of the lower being twice that of the upper fluid, and floats with its axis vertical, and its centre in the common surface of the fluids.

Supposing a small displacement to take place, 1st, in a vertical direction, 2ndly, about a horizontal line through its centre of gravity, shew that the times of the small oscillations will be respectively

$$\pi\sqrt{\left(\frac{2b}{g}\right)}, \text{ and } \pi\sqrt{\left(\frac{8}{5}\cdot\frac{b}{g}\frac{a^2+b^2}{a^2-b^2}\right)},$$

where a and b are the semi-axes of the generating ellipse.

7. A homogeneous solid floats completely immersed in a liquid, the density of which varies as the depth, with its centre of gravity at a depth h; prove that the time of a small vertical oscillation is

$$2\pi\sqrt{\frac{h}{g}}.$$

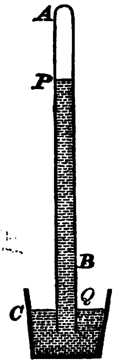
8. A lamina of uniform thickness, in the form of an isosceles right-angled triangle, has one of the acute angles fixed below the surface of a fluid, and rests with the side which is not immersed horizontal. Prove that the time of a small oscillation in its own plane is $2\pi\sqrt{\left(\frac{a}{g}\right)}$, where a is the length of each of the sides of the triangle.

CHAPTER VII.

PRESSURE OF THE ATMOSPHERE.

82. IF a glass tube, about three feet in length, having one end closed, be filled with mercury, and then inverted in a vessel of mercury so as to immerse its open end, it will be found that the mercury will descend in the tube, and rest with its upper surface at a height of about 29 inches above the surface of the mercury in the vessel: this experiment, first made by Torricelli, has suggested the use of the *Barometer*, for the purpose of measuring the atmospheric pressure.

The *Barometer*, in its simplest form, is a straight glass tube AB, containing mercury, and having its lower end immersed in a small cistern of mercury; the end A is hermetically sealed, and there is no air in the branch AB.



It is found that the height of the surface P of the mercury above the surface C is about 29 inches, and, as there is no pressure on the surface P, it is clear that the pressure of the air on C is the force which sustains the column of mercury PQ.

We have shewn that the pressure of a fluid at rest is the same at all points of the same horizontal plane; hence the pressure at C is equal to the pressure of the mercury at Q.

Let σ be the density of mercury, and Π the atmospheric pressure at C, then

$$\Pi = g\sigma PQ,$$

and the height PQ measures the atmospheric pressure.

On account of its great density, mercury is the most convenient fluid which can be employed in the construction of barometers, but the pressure of the air may be measured by using any kind of liquid. The density of mercury is about 13·568 times that of water, and therefore the height of the column of water in the water-barometer would be about $33\frac{3}{4}$ feet.

The density of mercury changes with the temperature, and σ must therefore be expressed as a function of the temperature.

Experiment shews that, for an increase of 1° centigrade, the expansion of mercury is $\frac{1}{5550}$th of its volume; hence if σ_t be the density at a temperature t°, and σ_0 at a temperature 0°,

$$\sigma_0 = \sigma_t\left(1 + \frac{t}{5550}\right) = \sigma_t(1 + \cdot00018018t);$$

$$\therefore\ \sigma_t = \sigma_0(1 - \theta t) \text{ if } \theta = \cdot00018018,$$

and $$\Pi = g\sigma_0(1 - \theta t)PQ.$$

By means of the formula, $\Pi = g\sigma_0(1 - \theta t)h$, the atmospheric pressure at any place can be calculated, making due allowance for the change in the value of g consequent on a change of latitude. It is found that this pressure is variable at the same place, with or without changes of temperature, and that in ascending mountains, or in any way rising above the level of the place, the pressure diminishes. This is in accordance with the theory of the equilibrium of fluids, for, in ascending, the height of the column of air above the barometer is diminished, and the pressure of the air upon C, which is equal to the weight of the superincumbent column of air, is therefore diminished, and the mercury must descend in the tube.

If then a relation be found between the height of the mercury and the height through which an ascent has been made, it is clear that by observations, at the *same* time, of the barometric columns at two stations, we shall be able to determine the difference of their altitudes.

We shall investigate a formula for this purpose; but it is first necessary to state the laws which regulate the pressures of the air and gases at different temperatures, and also the laws of the mixture of gases.

83. We have before stated the relation

$$p = k\rho\,(1 + \alpha t)$$

between the pressure, density, and temperature of an elastic fluid: it is deduced from the two following results of experiment:

(1) *If the temperature be constant, the pressure of air varies inversely as its volume. (Boyle's Law.)*

(2) *If the pressure remain constant, an increase of temperature of* 1° *C. produces in a mass of air an expansion* ·003665 *of its volume at* 0° *C. (Dalton's and Gay-Lussac's Law.)*

Hence, if p be the pressure and ρ_0 the density of air, at a temperature zero,

$$p = k\rho_0.$$

Suppose now the temperature increased to t, the pressure remaining the same: the conception of this may be assisted by considering the air to be contained in a cylinder in which a moveable piston fits closely, and has applied to it a constant force, so that an increase of the elastic force of the air would have the effect of pushing out the piston, until the equilibrium is restored by the diminution of density, and consequent diminution of pressure: we shall then have from the 2nd law,

$$\rho_0 = \rho\,(1 + \alpha t),$$

taking ρ as the new density and $\alpha = \cdot 003665$;

$$\therefore\ p = k\rho\,(1 + \alpha t).$$

If p', ρ' be the pressure and density of the same fluid at a temperature t',

$$p' = k\rho'\,(1 + \alpha t'),$$

and

$$\frac{p}{p'} = \frac{\rho}{\rho'}\,\frac{1 + \alpha t}{1 + \alpha t'}.$$

The quantity a is very nearly the same for gases of all kinds, but k has different values for different gases, and must of course be determined experimentally in every case*.

Absolute Temperature.

84. If we imagine the temperature of a gas lowered until its pressure vanishes, without any change of volume, we arrive at what is called the absolute zero of temperature, and absolute temperature is measured from this point.

Assuming t_0 to represent this temperature on the centigrade thermometer, we obtain, from the equation $1 + at_0 = 0$,

$$t_0 = -\frac{1}{a}, \quad = -273°.$$

In Fahrenheit's scale the reading for absolute zero is $-459°$.

The equations,
$$p = \kappa\rho\,(1 + at),$$
$$0 = \kappa\rho\,(1 + at_0),$$
lead to
$$p = \kappa\rho\alpha\,(t - t_0),$$
$$= \kappa\rho\alpha T,$$
if T be the absolute temperature.

Since ρV is constant, it follows that $\frac{pV}{T}$ is constant, and this law expresses, in the absolute scale, the relation between pressure, volume, and temperature.

The pressure of a mixture of different elastic fluids.

85. Consider two different gases, contained in vessels of which the volumes are V and V', and let their pressures and temperatures, p and t, be the same.

Let a communication be established between the two vessels, or transfer both the gases to a closed vessel, the volume of which is $V + V'$: it is found that, unless a chemical action take place, the two gases do not remain separate, but permeate each

* Methods of determining the value of a are described in Deschanel's *Natural Philosophy*, translated and edited by Professor Everett.

other until they are completely mixed, and that, when equilibrium is attained, the pressure and temperature are the same as before. From this important experimental fact we can deduce the following proposition.

If two gases having the same temperature be mixed together in a vessel, the volume of which is V, *and if the pressure of the two gases, alone filling the volume* V, *be* p *and* p′, *the pressure of the mixture will be* p + p′.

Suppose the two gases separated; let the gas, of which the pressure is p, have its volume changed, without any alteration of temperature, until its pressure becomes p'; its volume will be, by Marriotte's law, $\frac{p}{p'} V$.

Let the two gases be now mixed in a vessel, of which the solid content is

$$V + \frac{p}{p'} V, \text{ or } \frac{p+p'}{p'} V;$$

the pressure of the mixture will still be p', and the temperature will be unaltered. If the mixture be then compressed into a volume V, its pressure will become, by the application again of Marriotte's law, $p + p'$.

This result is obviously true for a mixture of any number of gases.

86. *Two volumes* V, V′ *of different gases, at pressures* p, p′ *respectively, are mixed together, so that the volume of the mixture is* U; *to find the pressure of the mixture.*

The pressures of the two gases, reduced to the volume U, are respectively

$$\frac{V}{U} p, \quad \frac{V'}{U} p',$$

and therefore, by the preceding article, the pressure of the mixture is

$$\frac{V}{U} p + \frac{V'}{U} p';$$

and if ϖ be this pressure, we have

$$\varpi U = pV + p'V'.$$

87. The laws and results of the preceding articles are equally true of *vapours*, the only difference between the *mechanical* qualities of vapours and gases, irrespective of their chemical characteristics, being that the former are easily condensed into liquids by lowering the temperature, while the latter can only be condensed by the application either of great pressure or extreme cold, or of a combination of both*.

88. If water be introduced into a space containing dry air, vapour is immediately formed, and it is found that the pressure and density of the vapour are dependent only on the temperature, and are quite independent of the density of the air, and indeed are exactly the same if the air be removed. If the temperature be increased or the space enlarged, an additional quantity of vapour will be formed, but if the temperature be lowered or the space diminished, some portion of the vapour will be condensed.

While a sufficient quantity of water remains, as a source from which vapour is supplied, the *space* will be always *saturated* with vapour, that is, there will be as much vapour as the temperature admits of; but if the temperature be so raised that all the water is turned into vapour, then for that, and all higher temperatures, the pressure of the vapour will follow the same law as the pressure of the air.

In any case, whether the space be *saturated* or not, if p be the pressure of the air, and ϖ of the vapour, the pressure of the mixture is $p+\varpi$.

89. The atmosphere always contains aqueous vapour, the quantity being greater or less at different times; if any portion

* Professor Faraday has succeeded in condensing a number of different gases; for instance, carbonic acid, hydrochloric acid, and others requiring a considerable pressure for the purpose, have been reduced to a liquid form. Oxygen, hydrogen, and nitrogen have not yet yielded, but there seems no reason to suppose, if a sufficient pressure can be applied, and a sufficient degree of cold obtained, that these gases will not follow the same law as those which have been liquefied. Such experimental results point to the general conclusion that all gases are merely the vapours of liquids of different kinds.

of the space occupied by the atmosphere be saturated with vapour, that is, if the density of the vapour be as great as it can be for the temperature, then any reduction of temperature will produce condensation of some portion of the vapour, but if the density of the vapour be not at its maximum for that temperature, no condensation will take place until the temperature is lowered below the point corresponding to the saturation of the space.

Formation of Dew. If any surface, in contact with the atmosphere, be cooled down below the temperature corresponding to the saturation of the space near it, condensation of the aqueous vapour will ensue, and the condensed vapour will be deposited in the form of *dew* upon the surface. The formation of dew on the ground depends therefore on the cooling of its surface, and this is in general greater and more quickly effected, when the sky is free from clouds, and when, consequently, the loss of heat by radiation is greater than under other circumstances.

The *Dew Point* is the temperature at which dew first begins to be formed, and must be determined by actual observation.

The pressure of vapour corresponding to its saturating densities for different temperatures must also be determined experimentally, and, if this be effected, an observation of the dew point at once determines the pressure of the vapour in the atmosphere. For if t' be the dew point, and p' the known corresponding pressure, then at any other temperature t above t' the pressure p is given by the equation

$$\frac{p}{p'} = \frac{1 + \alpha t}{1 + \alpha t'}.$$

Effect of compression or dilatation on the temperature of a gas.

90. It has been found by experience that, if a quantity of air be suddenly compressed its temperature is raised, and that, if the compression be of small amount, the relative increase of temperature is proportional to the condensation.

Thus, if the density be changed from ρ to ρ', the increase of temperature is proportional to $\dfrac{\rho' - \rho}{\rho}$.

If the air be allowed to dilate, its temperature is diminished according to the same law.

The reason for the suddenness of compression, or dilatation, is that no heat should be allowed to escape, or to be admitted. If the operation be performed in a non-conducting vessel, there is no necessity for rapidity of action.

91. *Thermal Capacity.*

The thermal capacity of a body is measured by the amount of heat required to raise the temperature of the body one degree.

The unit of heat which is usually employed is the quantity of heat required to increase by one degree the temperature of an unit mass of water, supposed to be between 0° C and 40° C.

Specific Heat.

The specific heat of a body is the thermal capacity of an unit of its mass.

Or, which is the same thing, it is the ratio of the amount of heat required to increase by 1° the temperature of the body to the amount of heat required to increase by 1° the temperature of an equal weight of water.

If an amount of heat δH produce in the unit of mass a change of temperature δt, the measure of the specific heat is $\frac{dH}{dt}$.

92. For gases it is necessary to consider two cases: (1) when the pressure remains constant, the gas being allowed to expand, (2) when the volume is constant.

In order to compare the specific heats in these two cases, suppose that, the pressure p remaining constant, the application of a small quantity of heat H changes the density from ρ' to ρ, and increases the temperature by τ; then

$$p = K\rho' T = K\rho (T + \tau).$$

Now let the air be rapidly compressed into its original volume;

$$\frac{\text{the increase of temperature}}{T} = \mu \frac{\rho' - \rho}{\rho}$$

$$= \mu \frac{\tau}{T},$$

and therefore the whole change of temperature produced by the heat H, when the volume is constant,

$$= \tau(1 + \mu) = \lambda\tau.$$

Hence, in order to produce a change of temperature τ, when the volume is constant, the amount of heat required is $\frac{H}{\lambda}$, and consequently,

$$\frac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}} = \frac{H}{\frac{H}{\lambda}} = \lambda.$$

This quantity λ is found experimentally to be constant for all simple gases, its value being approximately 1·408.

93. *A mass of air being suddenly compressed or dilated, it is required to find the changes of pressure and temperature.*

Supposing, which comes to the same thing, that the compression is effected in such a manner that no heat is lost or gained, let p, ρ, T be the pressure, density, and absolute temperature at any stage of the process, and let δT be the change of temperature due to the change $\delta\rho$ in ρ.

Then
$$\frac{\delta T}{T} = \mu \frac{\delta\rho}{\rho},$$

and therefore, since
$$p = K\rho T,$$

$$\frac{dp}{d\rho} = KT + K\mu T,$$

$$\frac{1}{p}\frac{dp}{d\rho} = \frac{1 + \mu}{\rho} = \frac{\lambda}{\rho},$$

and $$\frac{p'}{p} = \left(\frac{\rho'}{\rho}\right)^\lambda,$$

taking p' and ρ' as the new pressure and density.

Also, if T' be the new temperature,

$$\frac{p'}{p} = \frac{\rho' T'}{\rho T};$$

$$\therefore \left(\frac{\rho'}{\rho}\right)^{\lambda-1} = \frac{T'}{T},$$

from which T' is determined.

The increase of temperature caused by the sudden compression of atmospheric air is a fact, as will appear subsequently, of great importance in the theory of sound.

Whole mass of the Earth's Atmosphere.

94. Some idea may be formed of the mass of air and vapour surrounding the earth by means of the barometer. Supposing the earth to be a sphere of radius r, and that the height of the barometric column, h, is the same at all points of its surface, the mass of the atmosphere is approximately equivalent to the mass $4\pi\sigma r^2 h$ of mercury.

Let ρ be the mean density of the earth;

then, the mass of the atmosphere : the mass of the earth

$$= 4\pi\sigma r^2 h : \rho \frac{4}{3} \pi r^3$$
$$= 3\sigma h : \rho r.$$

But, taking water as the standard substance, $\sigma = 13{\cdot}57$, and ρ has been found to be about 5·5; and, if we take 29·9 inches as an approximate value of h, it will be found that the ratio of the masses is somewhat less than the ratio of one to a million*.

* The observations on the motion of pendulums, made by the Astronomer Royal at the Harton Colliery in 1854, have thrown doubt on the accuracy of the value 5·5, which has been assumed, in Art. 94, as a measure of the mean density of the earth.

The value deduced from the Harton *Observations* is 6·566 with a probable error ± ·0182. *Phil. Trans.* 1856.

The height of the homogeneous atmosphere.

95. If the whole column of air had the same density throughout as at the surface, its height being l, and the height of the mercury being h, we should have

$$\sigma h = \rho l,$$

where ρ is the density of the air. It has been found that the ratio $\sigma : \rho$ is about 10462 : 1, and therefore, employing as before 29·9 as a value of h, it will be found that l is a little less than 5 miles.

Necessary limit to the height of the atmosphere.

96. It is clear that, since at a distance from the earth's surface its attraction diminishes, and the density and pressure of the air are therefore diminished, the above result is very far from the truth. A *limit* to the height can however be found from the consideration that, beyond a certain distance from the earth's centre, its attraction will be unable to retain the particles of air in the circular paths, which they must describe about the earth, in order to remain in a state of relative equilibrium.

At the equator the expression $\omega^2 r$, ω being the earth's angular velocity, is equal to $\frac{g}{289}$, and therefore, at a height z, the force necessary to retain a particle m of air in its circular motion is equal to $\frac{mg}{289}\frac{r+z}{r}$; the earth's attraction at the same height

$$= \frac{mgr^2}{(r+z)^2};$$

and the extreme height is given by the equation

$$\frac{r^2}{(r+z)^2} = \frac{r+z}{289\,r}$$

or

$$z = r\{\sqrt[3]{(289)} - 1\},$$

that is, z is a little greater than $5r$.

It is possible however that this height is considerably beyond the true height, for the temperature of the air has been found, by experiments made in balloons, to diminish with great

rapidity during an ascent, and it is therefore quite possible, that, at a height less than $5r$, the air may be liquefied by extreme cold, and its external surface would be, in that case, of the same kind as the surfaces of known inelastic fluids.

The determination of heights by the barometer.

97. Consider a vertical column of the atmosphere at rest under the action of gravity: at a height z let p be the pressure and ρ the density, and at a height $z+\delta z$, let $p+\delta p$ be the pressure.

If A be the area of the section of the column, the volume $A\delta z$ of air may be considered as in equilibrium under the action of the pressures pA and $(p+\delta p)\,A$, and of its weight $g\rho A\delta z$.

Hence we have $\qquad \delta p = -g\rho\delta z;$

and, if t be the temperature, $p = k\rho\,(1+at)$;

$$\therefore \text{ in the limit } \frac{k}{p}\cdot\frac{dp}{dz} = -\frac{g}{1+at},$$

we shall suppose t constant, and therefore

$$k\log p = -\frac{gz}{1+at} + C,$$

and, if p' be the pressure at a height z', we obtain

$$k\log\frac{p}{p'} = \frac{g\,(z'-z)}{1+at}.$$

Let h, h', be the observed heights of the barometer at two stations, the heights of which are z and z'; then, taking σ as the density of mercury at a temperature zero, and τ, τ', as the temperatures at the two stations,

$$p = g\sigma h\,(1-\theta\tau), \text{ and } p' = g\sigma h'\,(1-\theta\tau');$$

$$\therefore z'-z = \frac{k}{g}\,(1+at)\log\frac{h\,(1-\theta\tau)}{h'\,(1-\theta\tau')};$$

t may be taken as approximately equal to $\frac{1}{2}(\tau+\tau')$, and we thus have an equation from which the difference of the heights of the two stations can be calculated.

98. If however the heights above the earth's surface be considerable, it is necessary to take account of the variation of gravity at different distances from the earth's centre. We proceed then to an investigation of a more exact formula.

Let g be the measure of gravity at the level of the sea, and r the radius of the earth, then, at a height z, the attractive force is measured by

$$g\frac{r^2}{(r+z)^2},$$

and the equation of equilibrium is

$$dp = -g\frac{r^2}{(r+z)^2}\rho dz;$$

we have also $p = k\rho(1+\alpha t)$, and it is here important to observe that p is the sum of the pressures due to the air itself, and to the aqueous vapour which is mixed with it, so that, if ρ' be the density of the aqueous vapour, p is the sum of two quantities in the form

$$k\rho(1+\alpha t) + k'\rho'(1+\alpha t),$$

and therefore the quantity $k\rho$ in the above equation is the sum of the two $k\rho$, $k'\rho'$, corresponding respectively to the air and the aqueous vapour.

From the two equations above we obtain

$$k\frac{dp}{p} = -\frac{1}{1+\alpha t}\frac{gr^2dz}{(r+z)^2},$$

and, as before, we shall consider t constant, and equal to the mean of the temperatures at the two stations.

By integration

$$k\log p = \frac{1}{1+\alpha t}\frac{gr^2}{r+z} + C,$$

$$\text{and } \therefore\ k\log\frac{p'}{p} = \frac{gr^2(z-z')}{(1+\alpha t)(r+z)(r+z')} \quad \text{..........} \ (1).$$

Let h, h', be the observed heights of the mercury, and τ, τ', the temperatures, as before; then, since the force of gravity at a height z is measured by the quantity $\frac{gr^2}{(r+z)^2}$, we have

$$p=\frac{gr^2}{(r+z)^2}\sigma h\,(1-\theta\tau),$$

$$p'=\frac{gr^2}{(r+z')^2}\sigma h'\,(1-\theta\tau'),$$

$$\frac{p'}{p}=\left(\frac{r+z}{r+z'}\right)^2\frac{1-\theta\tau'}{1-\theta\tau}\frac{h'}{h}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and therefore, observing that θ is a very small quantity,

$$z-z'=\frac{k(1+at)(r+z)(r+z')}{\mu gr^2}\{\log_{10}\frac{h'}{h}+2\log_{10}\frac{r+z}{r+z'}-\mu\theta(\tau'-\tau)\},$$

where $\mu=\log_{10}e=\cdot4342945$.

From this formula, if z' be known, the value of z can be calculated.

If the lower station be nearly at the level of the sea, $z'=0$, and

$$z=\frac{k(1+at)}{\mu g}\left(1+\frac{z}{r}\right)\{\log_{10}\frac{h'}{h}+2\log_{10}\left(1+\frac{z}{r}\right)-\mu\theta(\tau'-\tau)\}\ \ldots\ (3).$$

99. In the preceding investigation we have taken no account of the variation of gravity at different parts of the earth's surface; but if g' be the measure of gravity at a place of which the latitude is λ', and g at a place of latitude λ, it has been found, (Poisson, Art. 628), that

$$\frac{g}{g'}=\frac{1-\cdot002588\cos 2\lambda}{1-\cdot002588\cos 2\lambda'}:$$

the value of g obtained from this equation, in which g' and λ' are supposed to be known, must be employed in the above formula.

If λ' be the latitude of Paris, the value of the quantity

$$\frac{k}{\mu g'}(1-\cdot002588\cos 2\lambda')\ \ldots\ldots\ldots\ldots\ldots\ (4),$$

is nearly 18336 French metres or about 60158·56 English feet*, and, representing this numerical quantity by c, the expression for z becomes

$$\frac{c(1+at)\left(1+\frac{z}{r}\right)}{1-\cdot002588\cos 2\lambda}\{\log_{10}\frac{h'}{h}+2\log_{10}\left(1+\frac{z}{r}\right)-\mu\theta(\tau'-\tau)\}\ \ldots\ldots\ (5).$$

* A French metre is 39·37079 inches.

The value of c may be obtained by direct calculation of the expression (4), and the calculated value is 18337·46 metres; it has been found however, by comparing the results of trigonometrical measurements with the results of the formula (5), that 18336 metres is a more accurate value of the coefficient.

In order to calculate z from the formula (5), an approximate value must be first obtained by neglecting $\frac{z}{r}$ in the right-hand member of the equation; if this approximate value be then employed in the same expression, a more accurate value will result, and the same process may, if necessary, be repeated.

100. Other corrections are however necessary in order to render the determination of heights by the barometer very exact in practice; the value of k for instance is modified by the fact that the density of aqueous vapour at a given temperature and pressure is less than the density of dry air under the same circumstances, and the proportion of aqueous vapour to dry air may be, and in general will be, different at the two stations.

Moreover, if the upper station be on the surface of the ground, the attraction of the portion of the earth which is above its mean level must be taken account of. The effect of this attraction is to increase the quantity $\frac{gr^2}{(r+z)^2}$ by $\frac{3gz}{4r}$, so that, at a height z, the force of gravity is measured by

$$\frac{gr^2}{(r+z)^2}+\frac{3gz}{4r},$$

or, approximately, $g\left\{1-\frac{5z}{4r}\right\}$, (Poisson, *Mécanique*, Art. 629); the equation for p will be in this case

$$dp=-g\left\{1-\frac{5z}{4r}\right\}\rho dz,$$

and therefore, if the lower station be at the level of the sea,

$$k(1+at)\log\frac{p'}{p}=gz\left(1-\frac{5}{8}\frac{z}{r}\right)$$

or

$$z=\frac{k(1+at)}{g}\left(1+\frac{5}{8}\frac{z}{r}\right)\log\frac{p'}{p}.$$

In place of the equation (2) we shall have

$$\frac{p'}{p} = \left(1 + \frac{5}{4}\frac{z}{r}\right)\frac{1 - \theta\tau'}{1 - \theta\tau}\frac{h'}{h},$$

and the final equation for z will be obtained by substituting in (5),

$$1 + \frac{5}{8}\frac{z}{r} \text{ for } 1 + \frac{z}{r}, \text{ observing that } \log\left(1 + \frac{5}{4}\frac{z}{r}\right)$$

$$\text{is approximately equal to } 2\log\left(1 + \frac{5}{8}\frac{z}{r}\right).$$

When $\frac{z}{r}$ is very small, it may be neglected in the formula (5). It has however been found in practice that the results are rendered more accurate, for such cases, by employing, as the value of c, 18393 metres. (Duhamel, p. 259.)

101. In the preceding articles we have supposed the temperature of the air to be constant through the whole of the vertical space between the two stations; if however the difference between the heights be very great, a considerable error may be thus introduced, and formulæ have therefore been constructed in which account is taken, on various hypotheses, of the variation of atmospheric temperature. A formula of this kind is given in Lindenau's Barometric Tables, constructed on the supposition that the temperature diminishes in harmonic progression through a series of heights increasing in arithmetic progression.

It must also be noticed that we have assumed the temperature of the mercury in the barometer to be the same as that of the air surrounding it; but in some cases, as for instance when observations are made in a balloon, the barometer may not remain long enough in the same place to acquire the temperature of the air round it. The temperature of the mercury can, however, be observed by a thermometer the bulb of which is placed in the cistern of the barometer, and the temperatures so obtained must be employed in the equation (2) of Art. (98).

102. The two following problems are illustrative of the principles of this chapter.

(1) *A piston without weight fits into a vertical cylinder, closed at its base and filled with atmospheric air, and is initially at the top of the cylinder; water being poured slowly on the top of the piston, find how much can be poured in before it will run over.*

Let a be the height of the cylinder, and z the depth to which the piston will sink; then in the position of equilibrium the pressure of the air in the cylinder is $\Pi + g\rho z$, where Π is the atmospheric pressure, and ρ the density of water: but

$$\text{this pressure} : \Pi :: a : a - z;$$

$$\therefore \frac{\Pi a}{a - z} = \Pi + g\rho z.$$

Let h be the height of the water-barometer,

$$\therefore \Pi = g\rho h;$$

$$ha = (a - z)(h + z),$$

and $$z = 0 \text{ or } a - h.$$

Unless then the height of the cylinder is greater than h, no water can be poured in, for, even if the piston be forced down and water then poured on it, the pressure of the air beneath will raise the piston.

The negative solution, when $a < h$, can however be explained as the solution of a different problem leading to the same algebraic equation. Suppose the cylinder to be continued above the piston, and let it be required to raise the piston through a space z by a force which shall be equal to the weight of the cylindrical space z of water.

This leads to the equation

$$\frac{\Pi - g\rho z}{\Pi} = \frac{a}{a + z},$$

$$\text{or } z = h - a.$$

(2) *To determine the motion of a balloon on the supposition that the mass of air displaced by it in any position is homogeneous, and that the temperature throughout is constant.*

Let z be the height of the centre of gravity of the balloon, m its mass, V its volume, and ρ the density of the air at the height z; then the equation which determines the motion is

$$m\frac{d^2z}{dt^2} = g'\rho V - mg',$$

where $$g' = g\frac{r^2}{(r+z)^2}.$$

But from the equations $dp = -g'\rho dz$ and $p = k\rho$, we obtain

$$p = \Pi\epsilon^{-\frac{grz}{k(r+z)}},$$

and therefore

$$m\frac{d^2z}{dt^2} = \frac{\Pi Vgr^2}{k(r+z)^2}\epsilon^{-\frac{grz}{k(r+z)}} - mg\frac{r^2}{(r+z)^2};$$

from which, putting $m = \sigma V$, multiplying by $2\frac{dz}{dt}$, and integrating,

$$\sigma\left(\frac{dz}{dt}\right)^2 = C - 2\Pi\epsilon^{\frac{-grz}{k(r+z)}} + \frac{2\sigma gr^2}{r+z};$$

initially $$0 = C - 2\Pi + 2\sigma gr,$$

$$\therefore\ \sigma\left(\frac{dz}{dt}\right)^2 = 2\Pi\left\{1 - \epsilon^{\frac{-grz}{k(r+z)}}\right\} - \frac{2\sigma grz}{r+z}.$$

The greatest height of the balloon is given by putting

$$\frac{dz}{dt} = 0,$$

and, if the mean density of the balloon differ very little from that of the air, $\frac{z}{r}$ will be small, and an approximate value may be found.

EXAMPLES.

1. Two vessels contain air having the same pressure Π but different temperatures t, t'; the temperature of each being increased by the same quantity, find which has its pressure most increased.

If the vessels be of the same size, and the air in one be forced into the other, find the pressure of the mixture at a temperature zero.

2. The temperature of the air in an extensible spherical envelope is gradually raised t^0, and the envelope is allowed to expand till its radius is n times its original length; compare the pressure of the air in the two cases.

3. A cylindrical vessel, closed at both ends, and placed so that its axis is vertical, is half filled with mercury at a temperature $0^\circ C$, the remaining space being occupied by air at the same temperature. The expansion of mercury between the temperatures 0° and $100^\circ C$ being ·018 of its original volume, and that of air ·3665 of its original volume for the same pressure, shew that if the temperature be raised to $20^\circ C$ the pressure of the air will be increased in the ratio 1·0772 : 1.

4. If a given body lose in air, when the height of the barometric column is h, the m^{th} part of its weight, find what part of its weight it will lose when the height of the barometric column is h'.

5. The specific gravity of mercury compared with that of water at 68° is 13·568 and at 212° is 13·704. If the expansion of mercury between these points be $\frac{1}{69}$th of its volume at the lower temperature, find that of water between the same points.

6. A faulty barometer indicated 29·2 and 30 inches when the indications of a correct instrument were 29·4 and 30·3 inches respectively; find the length of tube which the air in the tube would fill under the pressure of 30 inches.

7. If a thermometer, plunged incompletely in a liquid whose temperature is required, indicate a temperature t, and τ be that of the air, the column not immersed being m degrees, prove that the

correction to be applied is $\frac{m(t-\tau)}{6840+\tau-m}$, $\frac{1}{6840}$ being the expansion of mercury in glass for 1° of temperature, assuming that the temperature of the mercury in each part is that of the medium which surrounds it.

8. A vertical barometer tube is constructed, of which the upper portion is closed at the top, and has a sectional area a^2, the middle portion is a bulb of volume b^3, and the lower portion has a section c^2, and is open at the bottom; the mercury fills the bulb and part of the upper and lower portions of the tube, and is prevented from running out below by means of a float against which the air presses; the upper part of the tube is a vacuum: find the change of position of the upper and lower ends of the mercurial column, due to a given alteration of the pressure of the atmosphere.

Shew also that, if the whole volume of the mercury in the instrument be c^2H, where H is the height of the barometer, the upper surface will be unaffected by changes of temperature.

9. A cylindrical diving-bell sinks in water until a certain portion V remains occupied by air, and in this position a quantity of air, whose volume under the atmospheric pressure was $2V$, is forced into it. Shew how far the bell must sink in order that the air may occupy the same space as in the first position.

Find also the condition that when the air is forced in at the first position no air may escape from beneath the bell.

10. Two equal closed cylinders both contain known quantities of water and air. One is placed above the other, and a communication made between the water in each. Find the amount which will flow from the upper to the lower before there is equilibrium.

Suppose the whole now introduced into a warm room, which way will the water flow?

11. A hollow cylinder containing air is fitted with an air-tight piston which when the cylinder is placed vertically is at a given height above the base; the cylinder being now inverted and placed vertically in a fluid sinks partly below the surface; find the position of equilibrium.

12. A vessel, in the form of the surface generated by the revolution about its axis of an arc of a parabola terminated by the vertex, is immersed, mouth downwards, in a trough of mercury; shew that the pressure of the air contained in the vessel varies inversely as the square of the distance of the vertex of the vessel from the surface of the mercury within it. Supposing the length of the axis of the vessel to be to the height of the barometer as 45 is to 64, find the depth of the surface of the mercury within the vessel, when the whole vessel is just immersed.

13. A piston without weight fits into a vertical cylinder, closed at its base and filled with air, and is initially at the top of the cylinder; if water be slowly poured on the top of the piston, shew that the upper surface of the water will be lowest when the depth of the water is $\sqrt{(ah)} - h$, where h is the height of the water-barometer, and a the height of the cylinder.

14. The barometer stands at 29·88 inches, and the thermometer is at the Dew Point: a barometer and a cup of water are placed under a receiver, from which the air is removed, and the barometer then stands at ·36 of an inch; find the space which would be occupied by a given volume of the atmosphere, if it were deprived of its vapour without changing its pressure or temperature.

15. A straight tube, closed at one end and open at the other, revolves with a constant angular velocity about an axis meeting the tube at right angles; neglecting the action of gravity, find the density of the air within the tube at any point.

16. A bent tube of uniform bore, the arms of which are at right angles, revolves with constant angular velocity ω about the axis of one of its arms, which is vertical and has its extremity immersed in water. Prove that the height to which the water will rise in the vertical arm is

$$\frac{\Pi}{g\rho}\left(1 - e^{-\frac{\omega^2 a^2}{2k}}\right),$$

a being the length of the horizontal arm, Π the atmospheric pressure, and ρ the density of water.

17. Prove that for rough purposes the difference of the logarithms of the heights of the barometer multiplied by 10000 gives the difference of the heights of two stations in fathoms.

18. Two bulbs containing air are connected by a horizontal glass tube of uniform bore, and a bubble of liquid in this tube separates the air into two equal quantities. The bubble is then displaced by heating the bulbs to temperatures t degrees and t' degrees: prove that, if the temperature of each bulb be decreased τ degrees, the bubble will receive an additional displacement which bears to the original displacement the ratio of $2\alpha\tau : 2 + \alpha(t + t' - 2\tau)$, where α is the coefficient of expansion.

19. An elastic spherical envelope is surrounded by air saturated with vapour; when the air within it is at a pressure of two atmospheres it is found that its radius is twice its natural length, and again the radius is three times its natural length when the envelope contains 77 times as much air as it would if open to the air; assuming that the tension at any point varies as the extension of the surface, prove that $\frac{1}{25}$ of the pressure of the air is due to the vapour it contains.

20. A piston of weight w rests in a vertical cylinder of transverse section k, being supported by a depth a of air. The piston rod receives a vertical blow P, which forces the piston down through a distance h: prove that

$$(w + \Pi k)\left\{h + a \log\left(1 - \frac{h}{a}\right)\right\} + \frac{gP^2}{2w} = 0,$$

Π being the atmospheric pressure.

CHAPTER VIII.

THE EQUILIBRIUM OF REVOLVING LIQUID, THE PARTICLES OF WHICH ARE MUTUALLY ATTRACTIVE.

103. If a liquid mass, the particles of which attract each other according to a definite law, revolve uniformly about a fixed axis, it is conceivable that, for a certain form of the free surface, the liquid particles may be in a state of relative equilibrium; since, however the resultant attraction of the mass upon any particle depends in general upon its form, which is unknown, a complete solution of the problem cannot be obtained.

For any arbitrarily assigned law of attraction, the question is one of purely abstract interest, and it is only when the law is that of gravitation that it becomes of importance, from its relation to one of the problems of physical astronomy.

We shall consider the fluid homogeneous, and confine our attention to two cases; in the first of these the attractive forces are supposed to vary directly as the distance, and, in the second, to follow the Newtonian law.

104. *A homogeneous liquid mass, the particles of which attract each other with a force varying directly as the distance, rotates uniformly about an axis through its centre of gravity; required to determine the form of the free surface.*

The resultant attraction on any particle is in the direction of, and proportional to, the distance of the particle from the centre of gravity; and if μ be a measure of the whole mass of fluid, μx, μy, μz, may represent the components of the attraction, parallel to the axis, on a particle of fluid about the point x, y, z.

Taking the origin at the centre of gravity, and axis of rotation as the axis of z, the equation of equilibrium is

$$dp = \rho \{(\omega^2 x - \mu x)\, dx + (\omega^2 y - \mu y)\, dy - \mu z dz\};$$

and therefore

$$p = C + \frac{\rho}{2} \{(\omega^2 - \mu)(x^2 + y^2) - \mu z^2\}.$$

At the free surface p is zero or constant, and the equation to the free surface is

$$\left(1 - \frac{\omega^2}{\mu}\right)(x^2 + y^2) + z^2 = D,$$

the constant D depending upon ω, and upon the mass of the fluid.

If $\omega^2 < \mu$, the free surface is a spheroid which becomes more oblate as ω increases, and when $\omega^2 = \mu$, the free surface consists of two planes; to render this possible we may conceive the fluid enclosed within a cylindrical surface, the axis of which coincides with the axis of rotation.

When $\omega^2 > \mu$, the free surface is a hyperboloid of two sheets, which for a certain value (ω') of ω becomes a cone, the fluid filling the space between the cone and the cylinder. Taking account of the volume of the fluid, the value of ω' can be determined by putting $D = 0$, since the pressure in this case vanishes at the origin.

If $\omega > \omega'$, the surface is a hyperboloid of one sheet, which, as ω increases, approximates to the form of a cylinder, and it is therefore necessary, for large values of ω, to conceive the containing cylinder closed at its ends.

The results of this article, it may be observed, are equally true of heterogeneous fluid, whatever be the law of variation of density in the successive strata.

105. *A mass of homogeneous liquid, the particles of which attract each other according to the Newtonian law, rotates uniformly, in a state of relative equilibrium, about an axis through its centre of gravity; required to determine a possible form of the surface.*

For the reason previously mentioned a direct solution of this problem cannot be obtained, but it can be shewn that an oblate spheroid is a possible form of equilibrium.

Let the equation to the spheroid be

$$\frac{z^2}{c^2}+\frac{x^2+y^2}{c^2(1+\lambda^2)}=1,$$

the axis of rotation being the axis of z.

Then the resultant attractions, towards the origin, on a particle at the point (x, y, z) will be represented by

$$X=\frac{2\pi\rho x}{\lambda^3}\{(1+\lambda^2)\tan^{-1}\lambda-\lambda\},$$

$$Y=\frac{2\pi\rho y}{\lambda^3}\{(1+\lambda^2)\tan^{-1}\lambda-\lambda\},$$

$$Z=\frac{4\pi\rho z}{\lambda^3}\{\lambda-\tan^{-1}\lambda\}(1+\lambda^2),$$

parallel, respectively, to the axes*.

We have then for the surfaces of equal pressure, putting ϵ for

$$\frac{\omega^2}{4\pi\rho},$$

$$\{2\epsilon\lambda^3+\lambda-(1+\lambda^2)\tan^{-1}\lambda\}(xdx+ydy)$$
$$+2(\tan^{-1}\lambda-\lambda)(1+\lambda^2)zdz=0.$$

But, from the equation to the spheroid,

$$xdx+ydy+(1+\lambda^2)zdz=0,$$

and, as these equations must be identical,

$$2\epsilon\lambda^3+\lambda-(1+\lambda^2)\tan^{-1}\lambda=2(\tan^{-1}\lambda-\lambda);$$

an equation the roots of which determine the possible values of λ.

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* These expressions will be found in Laplace's *Mécanique Céleste*, Poisson's *Mécanique*, Duhamel's *Mécanique*, and Todhunter's *Statics*. In the last named, the equation to the spheroid is $\frac{x^2+y^2}{a^2}+\frac{z^2}{a^2(1-e^2)}=1$, but the expressions used in the text will result from the expressions there given by putting $1-e^2=\frac{1}{1+\lambda^2}$.

By the use of λ, irrational quantities are avoided.

It may be written

$$\frac{3\lambda+2\epsilon\lambda^3}{3+\lambda^2}-\tan^{-1}\lambda=0, \qquad \text{.......................} \ (\alpha),$$

and the question is reduced to the discussion of the roots of this equation.

For this purpose consider the curve

$$y=\frac{3x+2\epsilon x^3}{3+x^2}-\tan^{-1}x; \qquad \text{....................} \ (\beta).$$

The abscissæ of the points where this curve cuts the axis will be the values of λ required.

It must be observed that, in the equation (α), $\tan^{-1}\lambda$ is the least positive angle whose tangent is λ; we have therefore only to consider one branch of the curve (β).

If the signs of x and y be changed, the equation is unaltered; the curve is therefore the same in the compartment $-x$, $-y$, as in $+x$, $+y$, and it is sufficient to examine the nature of the positive portion of the branch.

When $x=0$, $y=0$, and as x increases from zero, y begins by being positive, and when x increases indefinitely, has always positive values; hence the curve cuts the axis of x in an even number of points, exclusive of the origin.

Again, $$\frac{dy}{dx}=\frac{2x^2\{\epsilon x^4+2(5\epsilon-1)x^2+9\epsilon\}}{(1+x^2)(3+x^2)^2},$$

$\frac{dy}{dx}$ is therefore zero at the origin (a point of inflection), and also at the points given by

$$\epsilon x^4+2(5\epsilon-1)x^2+9\epsilon=0 \qquad \text{..............} \ (\gamma).$$

If the values of x^2, obtained from this equation, be real, and positive, there will be a maximum and a minimum value of y; the former, corresponding to the smallest root, will evidently be positive, since y begins by being positive; if the latter, corresponding to the greatest root, be negative or zero, there will be two zero values of y or one only, and consequently two possible spheroidal forms of equilibrium, or one only.

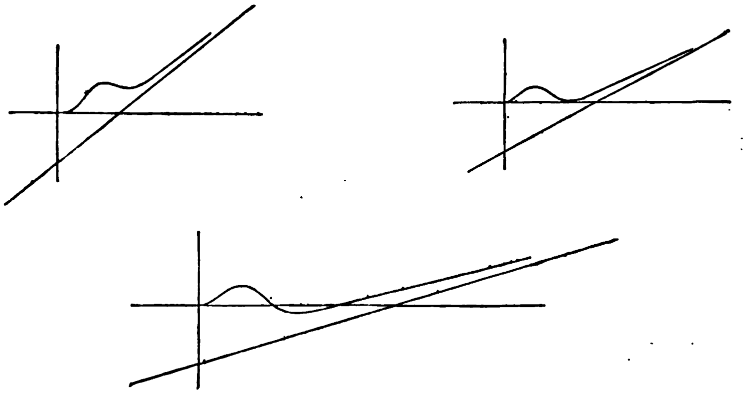
If the minimum value of y be positive, there will be no zero value of y; that is, the equilibrium of the fluid in the form of a spheroid is impossible.

106. The preceding investigation may be illustrated by tracing the curve (β) for different values of ϵ.

Putting $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1} \frac{1}{x}$, and expanding, we obtain

$$y = 2\epsilon x - \frac{\pi}{2},$$

as the asymptote of the branch of the curve under consideration, and the appended figures exemplify the different cases above mentioned.



Numerical Calculation.

107. *To calculate the limiting value of ω for which the spheroidal form is possible.*

The equation (γ) may have positive roots if $5\epsilon < 1$; moreover the values of x^2 will be real, and positive, if

$$(1 - 5\epsilon)^2 > 9\epsilon^2, \text{ or } 1 - 5\epsilon > 3\epsilon;$$

$$\text{i.e. } \epsilon < \frac{1}{8}.$$

The superior limiting value of ϵ can however be obtained very approximately from the condition that, in the extreme case of possibility, the minimum value of y is zero.

We have then $y = 0$ and $\frac{dy}{dx} = 0$, simultaneously.

Hence, substituting in (β) the value of ϵ obtained from (γ), and putting $y = 0$, we have

$$\frac{x(7x^4 + 30x^2 + 27)}{(x^2+1)(x^2+9)(x^2+3)} - \tan^{-1} x = 0,$$

or

$$\frac{x(7x^2+9)}{(x^2+1)(x^2+9)} - \tan^{-1} x = 0 \quad \ldots\ldots\ldots\ldots\ldots (\delta).$$

An approximate value of the positive root of this equation will be a value of x, which, substituted in (γ), will give approximately the superior limit of the value of ϵ.

Since $\omega^2 = 4\pi\rho\epsilon$ this determines the greatest possible rate of rotation consistent with the existence of a spheroidal form.

When ω is less than the limiting value thus obtained, there will be two spheroids, either of which will be a possible form of the rotating fluid.

108. *Approximate determination of the positive root of the equation*

$$\frac{x(7x^2+9)}{(x^2+1)(x^2+9)} - \tan^{-1} x = 0.$$

Denoting the first member by $f(x)$, it will be found that

$$f'(x) = \frac{8x^4(3-x^2)}{(x^2+1)^2(x^2+9)^2}:$$

this is positive from $x = 0$ to $x = \sqrt{3}$, and is afterwards negative; $f(x)$ therefore increases until $x = \sqrt{3}$, and then diminishes; and, since $f(0) = 0$, $f(x)$ begins by being positive.

By the use of the formulæ

$$\tan^{-1} 2 = \frac{\pi}{4} + \tan^{-1}\frac{1}{3},$$

$$\tan^{-1} 3 = \frac{\pi}{4} + \tan^{-1}\frac{1}{2},$$

it will be found without much difficulty that the root lies between 2 and 3, but the application of Newton's method with the value 2 as an approximate one shews that a closer limit will be convenient.

If then 2·5 be substituted we obtain, by the aid of the formula

$$\tan^{-1}(2{\cdot}5) = \tan^{-1}(2) + \tan^{-1}\frac{1}{12},$$

$$f(2{\cdot}5) = {\cdot}0025 \text{ approximately.}$$

Let $$x = 2{\cdot}5 + y,$$

then, approximately, $$y = -\frac{f(2{\cdot}5)}{f'(2{\cdot}5)},$$

but $$f'(2{\cdot}5) = -{\cdot}085, \text{ nearly;}$$

$$\therefore\ y = {\cdot}0293 \text{ and } x = 2{\cdot}5293.$$

The substitution of this value in (γ) will give

$$\epsilon = {\cdot}1123,$$

as the greatest possible value of ϵ or $\dfrac{\omega^2}{4\pi\rho}$.

Hence, when ω is such that $\epsilon < {\cdot}1123$, there are two spheroidal forms of equilibrium.

If ϵ is very small, one of the values of x (i.e. λ) will be very small and the other large, and therefore as ϵ decreases, the one spheroid becomes *very oblate* and approximates to a plane lamina, while the other approaches to the form of a sphere.

To find the small value of λ which satisfies the equation

$$\frac{3\lambda + 2\epsilon\lambda^3}{3 + \lambda^2} - \tan^{-1}\lambda = 0,$$

expand in ascending powers of λ, and we obtain

$$\lambda^2 = \frac{15\epsilon}{2} \text{ approximately.}$$

This gives a spheroid very slightly oblate, the ratio of its axes being $\sqrt{(1+\lambda^2)} : 1$, or very nearly $1 + \dfrac{15\epsilon}{4} : 1$.

The *large* value of λ is obtained by putting

$$\tan^{-1}\lambda = \frac{\pi}{2} - \tan^{-1}\frac{1}{\lambda},$$

and expanding in powers of $\frac{1}{\lambda}$, a process which gives

$$\lambda = \frac{\pi}{4\epsilon} - \frac{8}{\pi} + \text{terms involving positive powers of } \epsilon,$$

as an approximation.

109. *Application to the case of a fluid, the density of which is equal to the Earth's mean density.*

If r be the Earth's radius and ρ the mean density of the Earth,

$\frac{4}{3}\pi\rho r$ is the attraction at the surface of a sphere of fluid of the same radius as the Earth, and of density ρ.

Suppose for a moment that the Earth is homogeneous, and spherical,

then $\frac{4}{3}\pi\rho r$ measures the force of gravity at the pole.

But, since $\epsilon = \dfrac{\omega^2}{4\pi\rho}$, and therefore $3\epsilon = \dfrac{\omega^2 r}{\frac{4}{3}\pi\rho r}$,

$3\epsilon : 1$ :: difference of the measures of gravity at the pole and the equator : gravity at the pole (g).

Taking a second and a foot as the units of time and space, $g = 32$ approximately, $r = 4000 \times 1760 \times 3$, and it will be found that the time of rotation, $\dfrac{2\pi}{\omega}$, given by the limiting value ·1123 of ϵ, is a little more than $\dfrac{1}{10}$th of a day.

This then is the smallest time in which a homogeneous fluid mass, of density equal to the Earth's mean density, could rotate uniformly so as to be spheroidal in form.

110. The Earth, as is known by geodetic measurements, differs very slightly in its form from a sphere, and we can therefore apply our equations with great ease to the question of the homogeneity of the Earth, assuming it to have taken its present form when in a state of fluidity, or to be now a mass of fluid contained within a comparatively thin crust.

It has been found by observation, that for the Earth the ratio $\omega^2 r : g$ is about 1 : 289, and we have therefore

$$3\epsilon = \frac{1}{289}.$$

But from Art. (108), $\lambda^2 = \frac{15\epsilon}{2} = \frac{5}{579}$,

and the ratio of the axes of the spheroid

$$= 1 + \frac{\lambda^2}{2} : 1 = 232 : 231, \text{ nearly.}$$

This result does not accord with the facts obtained by actual measurement, which give 301 : 300 as an approximate value of the ratio.

The inference is that the Earth is not homogeneous.

111. The foregoing articles are taken chiefly from Laplace, *Mécanique Céleste*, Tome II.

It must be observed that the general problem of the form of a rotating fluid is not solved; all that is shewn being that, in certain cases, an oblate spheroid is a possible form of equilibrium.

If ω be such that $\epsilon > \cdot 1123$, it does not follow that equilibrium is impossible, but only that the spheroidal form cannot exist for that particular angular velocity.

If we put $-\lambda'^2$ for λ^2, taking λ'^2 as a positive quantity less than unity, the equation (γ) of Art. 105 becomes

$$\epsilon\lambda'^4 - 2(5\epsilon - 1)\lambda'^2 + 9\epsilon = 0,$$

or

$$\epsilon(1 - \lambda'^2)(9 - \lambda'^2) + 2\lambda'^2 = 0,$$

an equation which has no root less than unity.

From this it follows that a prolate spheroid is not a possible form of equilibrium*.

112. An important distinction has been pointed out by Poisson (Tome II. p. 547), between the surfaces of equal pressure in a fluid at rest under the action of extraneous forces, and in a fluid at rest, or revolving uniformly about a fixed axis, under the action of the mutually attractive forces of its particles.

Let ABC be the free surface, and DEF any surface of equal pressure; then, in the former case, the resultant force at any point of DEF is perpendicular to the surface at that point, and is unaffected by the existence of the fluid between ABC and DEF; this fluid could therefore be removed without affecting the equilibrium of the fluid mass bounded by DEF. In the latter case, the force at any point of DEF, although perpendicular to the surface at that point, is the resultant of the attractions of the mass of fluid contained by DEF, and of the mass contained between DEF and ABC; these two components of the resultant force are not necessarily perpendicular to the surface, and the fluid external to DEF cannot in general be removed without affecting the equilibrium of the remainder.

If, however, the fluid be homogeneous, and the particles attract each other according to the Newtonian law, so that the free surface may be spheroidal, the surfaces of equal pressure will be similar spheroids; and in this case, since the resultant attraction of an ellipsoidal shell on an internal particle is zero, the portion of fluid between ABC and DEF may be removed, provided the rate of rotation remain unaltered.

Moreover we have shewn, Art. (105), that for a given value of ω not exceeding a determined limit, there are two possible spheroidal forms: let ABC, the free surface, have one of these forms, and describe within the fluid mass a concentric spheroid, GHK, similar to the other spheroid; then the fluid between

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* *Méc. Céleste*, Tom. II. p. 59. The proof is also given in Pontécoulant's *Système du Monde*, Tom. II. p. 401.

ABC and *GHK* may be removed without affecting the fluid mass *GHK*.

The action of the shell upon a particle at a point P of the surface *GHK* is not perpendicular to the surface at P, but this action, combined with the attraction of the mass *GHK*, and the hypothetical force measured by $\omega^2 r$, is perpendicular to the surface, at P, of the spheroid passing through P, which is concentric with, and similar to, the surface *ABC*.

113. If a fluid mass be set in motion, about an axis through its centre of gravity, with an angular velocity such as to make the value of ϵ greater than the limit obtained in Art. (107), it does not follow that the fluid cannot be in equilibrium in the form of a spheroid, for it may be conceived that the mass will expand laterally with reference to the axis, taking a more flattened shape, until its angular velocity is so far diminished as to render the spheroidal form possible.

If the mass consist of *perfect* fluid, its form will oscillate through the spheroid of equilibrium, but if, as is the case in all known fluids, friction be called into play by the relative displacement of the particles, the oscillations will gradually diminish and at length a position of equilibrium will be attained. By D'Alembert's principle, the 'Angular momentum' of the system, relative to the axis, will remain constant, and this property of the motion enables us to determine the final angular velocity, and the form ultimately assumed*.

Considering the question generally, suppose the mass of fluid set in motion in any way, and then left to itself; the centre of gravity will be either at rest or moving uniformly in a straight line, and all we have to consider is the motion relative to the centre of gravity.

Draw through the centre of gravity the plane, in the direction of which the angular momentum is a maximum; then, however during the subsequent motion the fluid particles act on each other, this plane, which may be called the 'momental'

* The angular momentum of a system, relative to an axis, is the sum of the moments of the momenta of the several particles of the system about the axis.

plane, will remain fixed, and when the motion of the particles relative to each other has been destroyed by their mutual friction, the axis perpendicular to this plane will be the axis of rotation of the fluid mass in its state of relative equilibrium.

Let $2H$ be the given angular momentum of the system, and ω its ultimate angular velocity.

Taking c and $c\sqrt{(1+\lambda^2)}$ for the axes of the spheroid of equilibrium, and M for the mass, the expression for the angular momentum is $\frac{2}{5}Mc^2(1+\lambda^2)\,\omega$;

$$\therefore\ \frac{1}{5}Mc^2(1+\lambda^2)\,\omega = H;$$

we have also
$$\frac{4}{3}\pi\rho c^3(1+\lambda^2) = M,$$

and from these two equations, combined with the equation,

$$\frac{3\lambda+2\epsilon\lambda^3}{3+\lambda^2} - \tan^{-1}\lambda = 0 \ldots \text{Art. } (105),$$

the values of c, ω, and λ can be determined.

From the first two we obtain

$$\epsilon = \frac{\omega^2}{4\pi\rho} = \frac{25H^2\left(\frac{4}{3}\pi\rho\right)^{\frac{1}{3}}}{3M^{\frac{10}{3}}}(1+\lambda^2)^{-\frac{2}{3}},$$

$$= p\,(1+\lambda^2)^{-\frac{2}{3}}, \text{ suppose};$$

$$\therefore\ \frac{3\lambda+2p\lambda^3(1+\lambda^2)^{-\frac{2}{3}}}{3+\lambda^2} - \tan^{-1}\lambda = 0,$$

is the equation which determines λ.

The left-hand member of this equation is positive when λ is very small, and negative when λ is indefinitely large, and the equation has therefore a positive root; consequently, the fluid mass will at length attain a spheroidal form of equilibrium.

It can be shewn moreover that the equation has only one positive root, and therefore there is one spheroidal form, and

› only, towards which the oscillating fluid mass continually proximates.

This discussion is taken from the *Mécanique Céleste*, Tom. II. . 71, and from Pontécoulant's *Système du Monde*, Tom. II. ›. 409.

114. It was discovered by Jacobi that an ellipsoid with three unequal axes is a possible form of relative equilibrium for a mass of rotating liquid.

The following proof of Jacobi's theorem is taken from a paper by Liouville in the *Journal de l'École Polytechnique*, Tome XIV.

Taking the axis of rotation for the axis of z, suppose, if possible, that the surface of the liquid is of the form given by the equation

$$\frac{x^2}{1+\lambda^2}+\frac{y^2}{1+\lambda'^2}+z^2=c^2 \text{..............} (1).$$

Then, if M be the mass of the liquid, the resultant attractions on a particle at the point (x, y, z) of the surface are respectively Ax, By, and Cz*,

where
$$A=\frac{3M}{c^3}\int_0^1 \frac{u^2du}{(1+\lambda^2u^2)H},$$

$$B=\frac{3M}{c^3}\int_0^1 \frac{u^2du}{(1+\lambda'^2u^2)H},$$

$$C=\frac{3M}{c^3}\int_0^1 \frac{u^2du}{H},$$

H representing the expression

$$\sqrt{(1+\lambda^2u^2)(1+\lambda'^2u^2)}.$$

The differential equation of the free surface is

$$(Ax-\omega^2x)\,dx+(By-\omega^2y)\,dy+Czdz=0,$$

and therefore, if the free surface be the ellipsoid (1),

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* See the *Mécanique Céleste*, Tome II., or Duhamel's *Cours de Méca*

$$(A - \omega^2)(1 + \lambda^2) = (B - \omega^2)(1 + \lambda'^2) = C \text{.........} (2).$$

Eliminating ω^2, we obtain

$$(1 + \lambda^2)(1 + \lambda'^2)(A - B) = C(\lambda'^2 - \lambda^2),$$

and, substituting for A, B, and C, this reduces to

$$(1 + \lambda^2)(1 + \lambda'^2)\int_0^1 \frac{(\lambda'^2 - \lambda^2)\, u^4 du}{H^3} = (\lambda'^2 - \lambda^2)\int_0^1 \frac{u^2 du}{H}.$$

Rejecting the solution $\lambda' = \lambda$, which leads to the case of an oblate spheroid, and transposing, we obtain

$$\int_0^1 \frac{u^2(1 - u^2)(1 - \lambda^2\lambda'^2 u^2)\, du}{H^3} = 0,$$

an equation which, if λ be assigned, determines λ'.

Assigning a positive value to λ^2, the left-hand member of the equation is positive if $\lambda' = 0$, and is negative if $\lambda' = \infty$; hence there is a positive value of λ'^2 which will satisfy the equation.

Moreover, from the equations (2),

$$\omega^2 = A - \frac{C}{1 + \lambda^2}$$

$$= \frac{3M}{c^3}\int_0^1 \frac{\lambda^2(1 - u^2)\, u^2 du}{(1 + \lambda^2)(1 + \lambda^2 u^2) H},$$

and ω^2 is therefore a positive quantity.

Hence it is completely established that an ellipsoid with three unequal axes, the smallest of which coincides with the axis of rotation, is a possible form of the free surface.

115. It was pointed out by Mr Todhunter, and demonstrated in the following manner, that the relative equilibrium of the rotating ellipsoid cannot subsist when the axis of rotation does not coincide with a principal axis.

Referred to the principal axes, let l, m, n, be the direction cosines of the axis of rotation, M any point (x, y, z) of the mass, and N the foot of the perpendicular from M upon the axis.

Then $$ON = lx + my + nz,$$

and, if $ON = v$, the co-ordinates of N are lv, mv, nv.

The acceleration $\omega^2 MN$, when resolved parallel to the axes, gives rise to the components

$$\omega^2 (x - lv),\ \omega^2 (y - mv),\ \omega^2 (z - nv);$$

therefore the differential equation of the free surface is

$$\{\omega^2(x-lv) - Ax\}\, dx + \{\omega^2(y-mv) - By\}\, dy + \{\omega^2(z-nv) - Cz\}\, dz = 0;$$

hence the form of the free surface is given by the equation,

$$\omega^2 (x^2 + y^2 + z^2) - \omega^2 (lx + my + nz)^2 - Ax^2 - By^2 - Cz^2 = \text{constant},$$

and this cannot represent an ellipsoid referred to its principal axes, unless two of the quantities l, m, n, vanish.

Mr Greenhill remarks that a particle of the liquid at the end of the axis of rotation will be at rest under the action of the attraction of the liquid alone, since the centrifugal force at that point vanishes.

Hence the attraction on the particle must be normal to the surface, which is only the case at the end of an axis.

CHAPTER IX.

THE TENSION OF FLEXIBLE SURFACES.

116. THE general problem of the equilibrium of flexible surfaces is considered by Lagrange, *Mécanique Analytique*, Tom. I., and also, more fully, by Poisson, *Mémoires de l'Institut*, 1812; it is proposed in this Chapter to discuss one class of the questions which arise out of the general case, those namely which have reference to the action of fluids upon flexible surfaces.

The pressure of a fluid at rest being normal to any surface with which it is in contact, we have, in fact, to consider the equilibrium of flexible surfaces at rest under the action of normal pressures, and of the tensions at their bounding lines.

For the sake of generality the term 'flexible surface' is employed as the representative of substances, such as cloth and thin paper, which do not offer any sensible resistance to bending, and which, when bent or twisted, do not tend to return to their original form. Perfectly flexible surfaces, whether extensible or inextensible, are therefore to be looked upon as inelastic.

In the following articles we shall suppose that the stress between any two portions of a flexible surface is wholly tangential to the surface.

Measure of Tension.

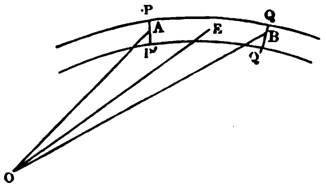
Conceive a flexible and inelastic surface, extensible or inextensible, in a state of tension, and let QPQ' be a small arc of the section through P made by a normal plane; then if $t \,.\, QQ'$ be the resultant action, perpendicular to QQ' in the tangent

plane, between the portions of surface bounded by the line QQ'; t is the measure of the tension at P; in other words, t is the rate of tension at P, or the force which would be exerted on a section of the substance, the length of which is unity, in the same state of tension throughout as the surface at P.

In general the stress between the portions of surface separated by QQ' will not be perpendicular to QQ', and will therefore be the resultant of the tension $t\,.\,QQ'$ and of a force $\tau\,.\,QQ'$ tangential to the curve QQ', τ being a quantity of the same kind as t and measured in the same way.

117. *A vessel in the form of a right circular cylinder, the curved surface of which is flexible, contains fluid; the axis of the cylinder being vertical, it is required to find the relation between the pressure and tension at any point.*

Let PQ' be a small portion of the surface contained between two planes perpendicular to the axis and two generating lines of the cylinder.



Let t be the horizontal tension and p the pressure, at any point of PQ, and suppose the element PQ' of the surface to be made rigid; then its equilibrium will be maintained by the normal pressure of the fluid, $pPP'\,.\,PQ$, the tangential forces tPP' and tQQ', and by the vertical tensions on PQ and $P'Q'$, if there be any tension in the vertical direction.

Hence, resolving the forces in the direction of the normal OE, drawn to the middle point E,

$$p\,.\,PP'\,.\,PQ = 2tPP'\sin\left(\frac{1}{2}POQ\right),$$

$$= 2tPP'\frac{1}{2}\frac{PQ}{r}, \text{ if } r \text{ be the radius,}$$

or $t = pr$.

118. *If fluid at rest under the action of given forces be contained in a cylindrical surface of any form, the tension at any point of a section perpendicular to the axis of the cylinder is the same.*

Let PQ', (figure, Art. 117), be an element of the surface, O the centre of curvature at A, t the tension at A, $t+\delta t$ at B, and $\delta\phi$ the angle between the tangents at A and B.

Also, let $\delta\psi$ be the inclination to OA of the direction of the fluid pressure on PQ', which must lie between OA and OB.

Then, resolving along the tangent at A,

$$(t+\delta t)\cos\delta\phi - t = pAB\sin\delta\psi,$$
$$= pr\delta\phi\sin\delta\psi,$$

if r be the radius of curvature at A.

Hence, ultimately, when $\delta\phi$ vanishes,

$$\frac{dt}{d\phi}=0,$$

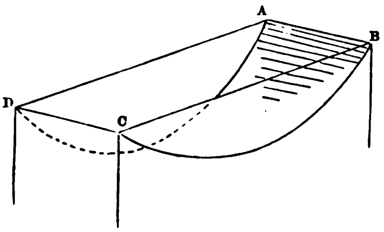
and, as this is the case at every point of the section, it follows that t is constant.

By resolving the forces in the direction OA, we shall obtain, as in the previous article, the relation

$$t=pr,$$

between the tension perpendicular to the generating line, the pressure, and the curvature, at any point of the surface.

Ex. *A rectangular piece of flexible and inextensible substance has its sides* AB, CD *fastened to the sides of a box, and its other sides fit the box closely, so that liquid is contained in it*



without escaping; required to determine the form of the curve BC.

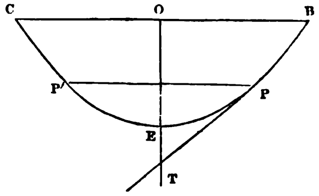
The surface formed is evidently cylindrical, and the tension at any point perpendicular to the direction of generating lines, and therefore constant throughout.

We have then $t = c$, and therefore $c = pr$, if r be the radius of curvature; but if x be the depth below the plane $ABCD$,

$$p = g\rho x,$$

and therefore $$c = g\rho x r.$$

Take the middle point of BC as origin, and OB as the axis of y;



then, for the arc BE, $r = \dfrac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$,

since $\dfrac{dy}{dx}$ decreases algebraically as x increases;

$$\therefore -\frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \frac{g\rho}{c}x;$$

integrating $$\frac{-\dfrac{dy}{dx}}{\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}} = C + \frac{g\rho}{2c}x^2.$$

If $OE = a$, $\dfrac{dy}{dx}$ is infinite and negative when $x = a$, and we obtain

$$\frac{-\frac{dy}{dx}}{\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}} = 1 + \frac{g\rho}{2c}(x^2 - a^2) = \frac{g\rho}{2c}(b^2 + x^2) \text{ suppose,}$$

or,
$$\frac{dy}{dx} = \mp \frac{g\rho\,(b^2 + x^2)}{\sqrt{\{4c^2 - g^2\rho^2(b^2 + x^2)^2\}}},$$

which is the differential equation to the curve, the sign being $-$ or $+$, according as x, y, are co-ordinates of P or P', and the constants being determined by the conditions that BC and the arc BEC are of given lengths.

Since $r = \frac{t}{p} = \frac{c}{p}$, it is clear that the curvature at each of the points B and C is zero.

Let $PTO = \phi$, then $\sin\phi = \frac{g\rho}{2c}(b^2 + x^2)$, and, if $AB = l$,

$$2lc\cos\phi = \text{the weight of the fluid above } PEP'$$

$$= 2g\rho lxy + \int_x^a 2g\rho yl\,dx,$$

$$\therefore \int_x^a 2y\,dx = \sqrt{\left\{\frac{4c^2}{g^2\rho^2} - (b^2 + x^2)^2\right\}} - 2xy,$$

an expression for the area PEP'.

Hence making $x = 0$, the area CEB

$$= \sqrt{\left(\frac{4c^2}{g^2\rho^2} - b^4\right)},$$

and the whole volume of fluid is

$$\frac{l}{g\rho}\sqrt{(4c^2 - g^2\rho^2 b^4)}.$$

If the curve be vertical at B and C, $\frac{dy}{dx} = 0$, when $x = 0$, and therefore $b = 0$, or

$$\frac{2c}{g\rho} - a^2 = 0;$$

$$\therefore a = \sqrt{\left(\frac{2c}{g\rho}\right)},$$

and the equation to the curve becomes

$$\frac{dy}{dx}=\frac{x^2}{\sqrt{(a^4-x^4)}}.$$

Let $2e$ be the length of the arc BEC, then

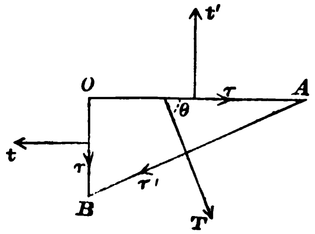
$$e=\int_0^a\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx=\int_0^a\frac{a^2dx}{\sqrt{(a^4-x^4)}},$$

an equation by which a, and therefore also c, is defined. The curve thus obtained is called the *Lintearia;* the investigation of its equation was first effected by James Bernoulli*.

119. Considering any flexible surface in equilibrium under the action of fluid pressures, the stress along any line, that is, the action between the contiguous portions of the surface bounded by that line, is in general oblique to the line, and is therefore represented by a tension t and a tangential action τ; we shall now shew that for any two directions, at right angles to each other, τ is the same, and that there are two directions for which τ vanishes.

Taking any small square element of the surface, which is ultimately plane, the tangential actions $\tau\delta s$ and $(\tau+\delta\tau)\delta s$ on a pair of opposite sides form ultimately a couple $\tau\delta s^2$, if δs be a side of the element; and, since this must be balanced by the other couple $\tau'\delta s^2$, if τ' be the tangential action in the direction at right angles, it follows that τ and τ' are equal.

Now take a small triangular element, OAB, right-angled at O, and represent the stresses as in the figure.



* The history of this problem is given in Walton's *Hydrostatical Problems,* p. 207. The Lintearia is the same as the Elastica, the curve formed by a bent elastic rod.

Resolving parallel to AB, we obtain

$$\tau' AB = tOB \sin\theta + \tau OB \cos\theta - t' OA \cos\theta - \tau OA \sin\theta,$$

$$\therefore\ 2\tau' = (t - t') \sin 2\theta + \tau \cos 2\theta,$$

and τ' vanishes when $\tan 2\theta = \dfrac{\tau}{t' - t}$, giving two directions at right angles.

120. If in the previous figure we assume that OA and OB are the directions of zero tangential action, and if we resolve in the directions parallel and perpendicular to AB, we shall obtain

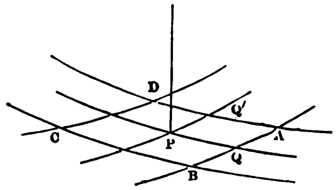
$$T = t \cos^2\theta + t' \sin^2\theta,$$

$$\tau' = (t' - t) \sin\theta \cos\theta.$$

The quantities t and t' are now the greatest and least, or the least and greatest tensions, and we shall therefore call them the Principal Tensions.

121. *A flexible surface of any form is exposed to the action of fluid; required to find the relation between the pressure, principal tensions, and the curvatures in the directions of these tensions, at any point.*

Let Q, Q', be points contiguous to P, on the lines of principal tension PQ, PQ', through P; draw normal planes through Q and Q', perpendicular to the lines, PQ, PQ', cutting the surface in the arcs AB, AD, and let BC, CD, be the arcs of section made by normal planes through contiguous points in $Q'P$, QP, produced.



The element BD is kept at rest by the tangential forces tAB, tCD, $t'AD$, $t'BC$, and the normal force, $p.AB.BC$.

Let r, r', be the radii of curvature at P of the curves PQ, PQ'; then, resolving along the normal at P, we have ultimately

$$p \,.\, AB \,.\, BC = 2tAB\frac{\frac{1}{2}AD}{r} + 2t'BC\frac{\frac{1}{2}AB}{r'},$$

and
$$\therefore\ p = \frac{t}{r} + \frac{t'}{r'}.$$

If the nature of the surface be such that $t' = t$, the above equation is

$$\frac{p}{t} = \frac{1}{r} + \frac{1}{r'},$$

or, if $z = f(x, y)$ be the equation to the surface,

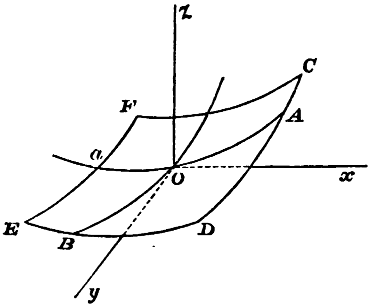
$$\frac{p}{t} \cdot \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{3}{2}}$$
$$= \left\{1 + \left(\frac{dz}{dy}\right)^2\right\}\frac{d^2z}{dx^2} - 2\frac{dz}{dx}\frac{dz}{dy}\frac{d^2z}{dxdy} + \left\{1 + \left(\frac{dz}{dx}\right)^2\right\}\frac{d^2z}{dy^2};$$

the equation obtained by Lagrange and Poisson.

122. If the directions of t and t' are not those of principal tensions the tangential action will appear in the equation.

Taking any point O on the surface, two directions OA, OB at right angles to each other, let t, t' be the tensions in these directions, and T', T the tangential actions in the same directions.

Oz being the normal at O, draw four planes parallel to, and very near to, the normal planes AOz, BOz, cutting the surface in CD, DE, EF, FC.



Then, ultimately, the tangential actions, $T.CD$ and $T.EF$ on CD and EF are equal and opposite, as are also those on ED and CF.

Hence, by taking moments about OZ, it appears that $T=T'$, as in Art. 119.

If θ be the inclination to the plane xy of the tangent at A to the curve CD,

$$\tan\theta = \frac{d^2z}{dxdy}.OA,$$

and similarly at the point a,

$$\tan\theta' = \frac{d^2z}{dxdy}(-Oa).$$

Hence the sum of the actions $T.CD$ and $T.EF$ in the direction Oz

$$= T.CD\frac{d^2z}{dxdy}OA - T.EF\frac{d^2z}{dxdy}(-Oa) = T.CD.DE.\frac{d^2z}{dxdy},$$

and a similar term arises from the action T'.

Resolving along Oz, we now obtain

$$p.CD.DE = 2tCD\frac{OA}{r} + 2t'DE\frac{OB}{r'} + 2T.CD.DE\frac{d^2z}{dxdy},$$

and $$\therefore\ p = \frac{t}{r} + \frac{t'}{r'} + 2T\frac{d^2z}{dxdy}\text{*}.$$

123. If we imagine a surface of such a nature that the tension at any point is always perpendicular to a line of division through that point, it can be shewn that the tension at any point is the same in every direction.

Let a small triangular portion of the surface be supposed rigid; then the equilibrium in the tangent plane is entirely determined by the tensions of the sides of the triangle, for the

* The general question of the equilibrium of flexible surfaces is discussed in a paper contained in the *Quarterly Journal of Mathematics*, Vol. IV. 1860.

See also a paper in the same journal, Vol. VIII. 1867, on the equilibrium of a spherical envelope, by Professor Clerk Maxwell.

tangential impressed forces, if there be any, will ultimately vanish in comparison with the tensions; and since these tensions are perpendicular to the sides, they must be in the ratio of their lengths, and therefore the measures of tension in all directions are the same.

Further, the tension will be the same over the surface, for, if a small rectangular element be considered, the tensions on the opposite sides must be equal.

The conception of such a surface is of the same nature as the conception of a perfectly rigid body or of a perfect fluid; nevertheless we obtain approximate specimens in the case of liquid films, such as soap-bubbles, or the films which may be seen in a clear glass bottle containing liquid which has been shaken about. Such films can also be practically obtained by dipping a wire frame in a solution of soap and water and slowly drawing it out.

For instance, a circular wire immersed horizontally, and slowly lifted, will produce a film in the form of a surface of revolution, such as would be produced by the revolution of a catenary about its directrix.

To illustrate by another example, imagine a liquid film at first rectangular and bounded by the sides of a cylindrical wet tube and by diametral wires; turn one of the wires round the axis of the tube, and the film will then settle into a form satisfying the condition,

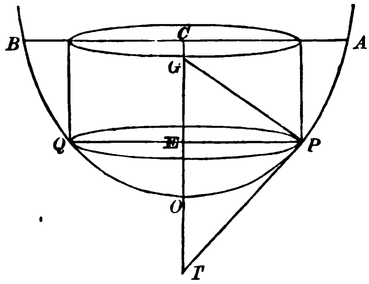
$$\frac{1}{r} + \frac{1}{r'} = 0,$$

the atmospheric pressure being the same on both sides.

A helicoidal form will obviously satisfy all the conditions of this case.

124. *A vessel, formed of flexible and inextensible material, is in the form of a surface of revolution, and is held with its axis vertical, and filled with homogeneous liquid: it is required to determine the principal tensions at any point.*

Let O be the lowest point of the vessel, and take O for the origin.



Measure z vertically upwards, and let PEQ be any horizontal section, the upper rim being ACB, which is supposed to be fixed.

At all points of the horizontal section PQ, the tensions are evidently the same.

Let t be the meridional tension, i.e. the tension at P, in direction of the tangent at P to the curve AP, and t' the horizontal tension at P; these are the principal tensions.

The vertical resultant of the tension t along the section PQ counterbalances the resultant vertical pressure on the surface POQ; hence, if

$$OE = z, \quad EP = x, \text{ and angle } PTO = \theta,$$

$$2\pi x t \cos\theta = \int_0^z g\rho\pi x'^2\, dz' + g\rho\pi x^2 (c - z),$$

if $OC = c$, and x', z' be co-ordinates of any point in the arc OP.

This equation determines t, and t' is given by the equation

$$\frac{t}{r} + \frac{t'}{r'} = p, \text{ Art. } 121^*,$$

where $p = g\rho (c - z)$.

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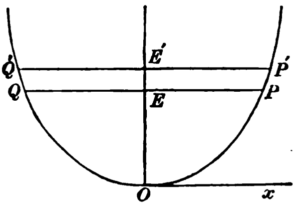
* This equation may also be obtained, for this case, by taking a small element bounded by lines of curvature, that is by meridians and horizontal circles; it will be necessary to employ Meunier's theorem, and to observe that the osculating planes of lines of curvature are not generally normal planes.

It will be observed that r is the radius of curvature of the curve AP at P, and that r', the radius of curvature of the perpendicular normal section, is the normal PG.

125. A more general proposition is the following:

A flexible vessel, in the form of a surface of revolution, is subject to fluid pressure, such that it is the same at all points of the same circular section; it is required to determine the principal tensions at any point.

Let PEQ, $P'E'Q'$ be two consecutive circular sections, and let t be the meridional tension at P.



If $OP = s$, the resultant tension, parallel to the axis, on the circle PQ,

$$= 2\pi xt \frac{dz}{ds};$$

$\therefore$ the resultant tension, parallel to Oz, on $P'Q'$

$$= 2\pi \left\{ xt \frac{dz}{ds} + \frac{d}{ds}\left(xt \frac{dz}{ds}\right)\delta s \right\}, \text{ if } PP' = \delta s.$$

The difference of these two counterbalances the resultant pressure, parallel to Oz, on the strip of surface between the circles PQ, $P'Q'$, which is equal to

$$p \,.\, 2\pi x \delta s \frac{dx}{ds},$$

if p be the pressure at any point of the circle PQ;

$$\therefore \frac{d}{ds}\left(xt \frac{dz}{ds}\right) = px \frac{dx}{ds},$$

and p being a given function of z, and therefore of s, this equation determines the tension t.

As before, t' is given by the equation

$$\frac{t}{r} + \frac{t'}{r'} = p.$$

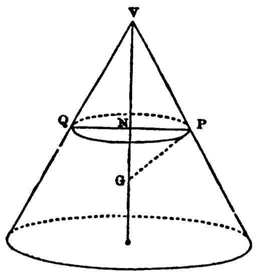
Taking a small element $PP'R'R$ bounded by meridian arcs PP', RR', and by circular arcs PR, $P'R'$, and resolving in direction of the meridian bisecting PR and $P'R'$, we obtain

$$t' = \frac{d}{dx}(xt),$$

a result which of course can also be obtained by eliminating p from the preceding equations.

126. Ex. *A conical perfectly flexible and elastic bag attached, mouth downwards, by the rim to a horizontal plane, and filled with liquid by a small hole at the apex, has, when at rest, the figure of a right circular cone; find the equation to the figure it will assume when detached and the liquid let out, neglecting its weight.*

Let t be the tension at P in the direction perpendicular to the generating line VP, t' the tension in the direction VP, and 2α the vertical angle of the cone.



Then $$p = \frac{t}{r} + \frac{t'}{r'} \text{ gives, if } VN = x,$$

$$g\rho x = \frac{t}{PG} = \frac{t}{x \tan\alpha \sec\alpha},$$

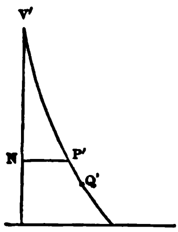
or $$t = g\rho x^2 \tan\alpha \sec\alpha.$$

But $2\pi PNt' \cos\alpha =$ the resultant vertical pressure on VPQ

$$=\frac{2}{3}g\rho\pi x^3 \tan^2 \alpha;$$

$$\therefore\ t' = \frac{1}{3} g\rho x^2 \tan \alpha \sec \alpha.$$

Let $V'P'Q'$ be the generating curve of the surface of revolution into which the surface forms itself after the liquid has been let out, $V'N = \xi$, $P'N = \eta$, P' corresponding to the point P.



If $P'Q' = \delta s$, a small arc of the curve,

$$\delta x \sec \alpha = \delta s \left(1 + \frac{t'}{\lambda'}\right),$$

and $$x \tan \alpha = \eta \left(1 + \frac{t}{\lambda}\right),$$

taking the modulus of elasticity different in the two directions. Taking account of the values of t and t' obtained above, x can be eliminated between these two equations, and the relation between ξ and η will result.

From the first equation, putting $\dfrac{g\rho \tan \alpha \sec \alpha}{3\lambda'} = \dfrac{1}{a^2}$,

$$\frac{ds}{dx} \cos \alpha = \frac{1}{1 + \dfrac{x^2}{a^2}};$$

$$\therefore\ \frac{s}{a} \cos \alpha = \tan^{-1} \frac{x}{a}, \text{ measuring } s \text{ from } V',$$

or $$\frac{x}{a} = \tan \left(\frac{s}{a} \cos \alpha\right).$$

Substituting this expression for x in the second equation, we obtain

$$a \tan \alpha \tan \left(\frac{s}{a} \cos \alpha\right) = \eta \left\{1 + \frac{g\rho a^2 \tan \alpha \sec \alpha}{\lambda} \tan^2 \left(\frac{s}{a} \cos \alpha\right)\right\},$$

as the differential equation to the curve*.

---

* If $\lambda = \lambda'$, the equation is

$$a \tan \alpha = \eta \left\{ \cot\left(\frac{s}{a} \cos \alpha\right) + 3 \tan\left(\frac{s}{a} \cos \alpha\right) \right\}.$$

127. We have hitherto considered only laminæ of uniform thickness, but, in order to include cases in which the lamina is of variable thickness, a more general measure of the tension can be given.

Suppose a bar AB of any homogeneous material to support a weight W, and let κ be the area of the section of the bar; then the tension at the section through P supports W and the weight of the bar PB; and if $\tau\kappa$ is equal to the sum of these weights, τ is the measure of the tension at P per unit of area.

It will be seen that τ is one dimension lower than the t of Art. (116).

In fact, if e be the thickness of a flexible lamina at any point, the tension at which, measured in the usual way per unit of length of section, is t, we have

$$t\delta s = \tau e \delta s,$$

$$\text{or } t = e\tau.$$

128. The investigations of this chapter will not in general be applicable to surfaces which are inflexible, or of imperfect flexibility, but, if in any particular case the action between adjacent portions of a surface be wholly in the tangent plane, the relations obtained between the tension and the normal pressure will hold good.

For instance, if a vertical circular cylinder formed of any inflexible substance be filled with fluid, the action at any point will be wholly tangential and of the nature of tension.

EXAMPLES.

1. Supposing the cylinders of a Bramah's Press made of the same material, and the stress to be the same in each, what should be the ratio of the thicknesses of the cylinders?

2. A cylindrical vessel is formed of metal a inches thick, and a bar of this metal of which the section is A square inches, will just bear a weight W without breaking. If the cylinder be placed with its axis vertical, find how much fluid can be poured into it without bursting it.

3. A hollow cone, the vertex of which is downwards, is filled with water; find where the horizontal tension is greatest.

Also find where the tension in the direction of a generating line is greatest.

4. The top of a rectangular box is closed by an uniform elastic band, fastened at two opposite sides, and fitting closely to the other sides; the air being gradually removed from the box, find the successive forms assumed by the elastic band, and when it just touches the bottom of the box, find the difference between the external and internal atmospheric pressures.

5. An elastic tube of circular bore is placed within a rigid tube of square bore which it exactly fits in its unstretched state, the tubes being of indefinite length; if there be no air between the tubes and air of any pressure be forced into the elastic tube, shew that this pressure is proportional to the ratio of the part of the elastic tube that is in contact with the rigid tube to the part that is curved.

6. A vessel, formed of a thin substance, in the shape of a cone with its axis vertical and vertex downwards, is just filled with liquid and closed at the top. If it be made to rotate uniformly about its axis, find the principal tensions at any point.

7. A vertical cylindrical vessel, formed of flexible and inextensible material, is put into a square box, the width of which is less than the diameter of the cylinder, and water is then poured into the cylinder; find the tension at any depth.

8. An elastic spherical envelope whose natural radius is a, has air forced into it so that its radius becomes b; it is then placed under an exhausted receiver, and its radius increases to c; find the quantity of air forced in, assuming that the tension is proportional to the increase of surface.

9. A hemispherical bag, supported at its rim, is filled with water; the principal tensions at a depth x are in the ratio

$$x^2 + ax + a^2 : 2x^2 + 2ax - a^2.$$

Find also where the horizontal tension vanishes, and explain the circumstance of its being negative for a portion of the bag.

10. If the hemispherical bag be closed at the top by a rigid plane to which its rim is tied, and then inverted, shew that the principal tensions at a depth x, are in the ratio

$$3a - 2x \; : \; 9a - 4x.$$

11. A spherical envelope is just filled with liquid, which rotates uniformly about a diameter; neglecting gravity, prove that the principal tensions at an angular distance ϕ from the axis of rotation are

$$\frac{1}{8}\rho\omega^2a^3\sin^2\phi \text{ and } \frac{3}{8}\rho\omega^2a^3\sin^2\phi.$$

12. A cylindrical shell of finite thickness is formed of a material such that a bar, one square inch in section, can sustain a tension τ without giving way. If this shell be subjected to an internal fluid pressure ϖ, which is only just not sufficient to burst the cylinder, prove that $\varpi = \tau \log \frac{a}{b}$; where a and b are the external and internal radii of the shell.

13. Shew, from the equations of Art. 125, that, if t be equal to t' at every point, each is constant.

Shew also that, in general, if t be a maximum or a minimum, it will be equal to t'.

14. A small uniform flexible tube is inextensible in length, but the perimeter of any transverse section of it follows the ordinary law of extension of elastic strings; if it be filled with liquid and held with its axis vertical, shew that for some distance from the highest point it will appreciably coincide with the surface generated by a rectangular hyperbola revolving about its asymptote.

15. A flexible bag, in the form of a right circular cone, just filled with liquid, has the rim of its base fastened to a rigid plane, and the liquid is acted upon by repulsive forces from the centre of the base, varying as the distance; find the principal tensions at any point.

If an aperture be made in the rigid plane, fitted with a piston, and a blow be struck on the piston, find the principal impulsive tensions at any point.

16. If, in Art. 124, the vessel be a paraboloid, and if the principal tensions be equal at any point of the horizontal section through the focus, shew that the length of the axis is $\frac{5}{8}$ths of the latus-rectum.

17. Shew that if a light thread with its ends tied together form part of the internal boundary of a liquid film, the curvature of the thread at every point will be constant.

If the thread have weight, and if the film be a surface of revolution about a vertical axis, prove that, in the position of equilibrium, the tension of the thread is

$$\frac{l}{2\pi}\sqrt{\tau^2 - w^2},$$

l being its length, w its weight per unit length, and τ the tension of the film.

18. A circular wire in a vertical plane has a short piece of thread tied to two points on its circumference; if a soap-bubble film be applied to the wire so that the string and the lower segment of the wire are its boundaries, the thread will assume the form given by the differential equation

$$ydy = \sqrt{\alpha y^2 + \beta y + \gamma}\, dx,$$

where α, β, γ are constants depending on the weight of the string, the superficial tension of the film, and the relative co-ordinates of the ends of the string.

Point out, from the equation, how the curve degenerates into a circle and a catenary.

19. Two soap-bubbles are in contact; if r_1, r_2 be the radii of the outer surfaces, and r the radius of the circle in which the three surfaces intersect;

$$\frac{3}{4r^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{1}{r_1 r_2}.$$

20. If a frame of fine straight wire in the form of a tetrahedron be lowered into a solution of soap in water and drawn up again, there are found in certain cases plane films starting from the edges and meeting in a point. Shew that this is not a possible form of equilibrium for every tetrahedron, and that it is so if one face be an equilateral triangle and the others isosceles triangles whose vertical angles are each less than $\sec^{-1}(-3)$.

21. A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid pressure. Supposing that the tensions in the meridian and circular sections are regulated by Hooke's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x + A = \int \frac{\frac{py^2}{2} + B}{\sqrt{\left(\frac{\lambda y^2}{2a} - \lambda y + C\right)^2 - \left(\frac{py^2}{2} + B\right)^2}}\, dy,$$

where a is the original radius, λ one of the moduli of elasticity, and A, B, C, constants of integration.

22. A soap-bubble film is stretched between two circular wires the centres of which are in a line perpendicular to their planes; prove that a section of the film by a plane through the axis of revolution will be the common catenary.

Considering the case in which the circles are equal, shew that according to the distance of their planes there will be two positions of equilibrium or none, and that in the former case the equilibrium in one position will be stable, and in the other unstable.

What will take place if the wires be slowly drawn asunder until the equilibrium becomes impossible?

CHAPTER X.

THE EQUATIONS OF MOTION.

129. We have established, as a consequence of the definition of a fluid, that the pressure at any point of a fluid at rest is normal to any surface with which it may be in contact, and that this is true, whether the fluid be a perfect fluid, or a viscous fluid.

Hence it follows, (Art. 7), that the pressure at any point of a fluid at rest is the same in all directions.

We now proceed to shew that, for a perfect fluid, whether at rest or in motion, the pressure at any point is the same in all directions.

Conceive a small tetrahedron of the fluid solidified; then, by D'Alembert's principle, the pressures on the faces, the moving forces arising from external attractions, and forces equal and opposite to the effective moving forces, form a system in equilibrium. Suppose the tetrahedron indefinitely diminished; the extraneous forces, and the effective moving forces, vary as the cubes, while the pressures vary as the squares, of homologous lines, and therefore the former are, in the limit, evanescent compared with the latter. The tetrahedron is, therefore, ultimately in equilibrium under the action of the pressures only, and hence it follows, as in Art. (7), that the pressure is the same in every direction.

The motion of perfect fluids only will be considered in the present treatise.

130. In the discussion of fluid motion, two methods may be employed, one, which may be called the flux method, in which we observe the changes going on at a fixed point, or within a given fixed portion of space; and another, which

may be called the force method, in which we observe the changes of motion of a particular element of fluid.

We shall obtain the equations requisite for the determination of fluid motions from both these points of view.

131. In the flux method the velocity at any point is measured by the volume which passes across an area, equal to the unit of area, in the unit of time.

If the velocity be uniform the volume which flows across a given plane area in a given time is equal to the product of the area, the time, and the component, perpendicular to the area, of the velocity.

For, if v be the velocity, the volume which flows across an area A in the time t will form an oblique prism on the base A, of length vt, with its generating lines in the direction of motion.

Taking θ as the angle between the direction of motion and the normal to the area, the volume of this prism is equal to $Avt \cos\theta$,

$$\text{i.e. } A.t.v\cos\theta.$$

In this case the velocity at any point is a function of the time and of the co-ordinates of the points.

In the force method, if we take a, b, c as the co-ordinates of a particle of fluid at an assigned epoch, for instance, when $t = 0$, and x, y, z as the co-ordinates of the same particle at the time t, the velocity of the particle is a function of a, b, c, and t.

132. To find the acceleration of a particle of fluid which at the time t is at the point x, y, z, let u, v, w be the component velocities at the point, which are functions of x, y, z, and t.

At the time $t + \delta t$, the co-ordinates of the particle of fluid which at the time t was at the point x, y, z, are

$$x + u\delta t,\ y + v\delta t,\ z + w\delta t;$$

therefore, if $$u = f(x, y, z, t),$$

the component velocity $u + Du$ of the same particle of fluid at the time $t + \delta t$,

$$= f(x + u\delta t,\ y + v\delta t,\ z + w\delta t,\ t + \delta t),$$

and the acceleration = limit of $\dfrac{Du}{\delta t}$

$$= \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}.$$

Similarly, the accelerations parallel to y and z are

$$\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz},$$

$$\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}.$$

It should be remarked that if $F(x, y, z, t)$ be any function whatever of the position of a particle, the rate of change of the value of the function is

$$\frac{dF}{dt} + u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz}.$$

133. *The Equation of Continuity.*

A fluid, whether at rest or in motion, is always a continuous mass.

Fixing our attention upon any small parallelepiped, fixed in space within the fluid, we have to express the fact that the increase of mass within this parallelepiped, in any small interval of time, is due to the excess of the mass of fluid which has entered over that which has passed out.

Let x, y, z, be the co-ordinates of one angular point, P, and $x + \alpha,\ y + \beta,\ z + \gamma$, of the opposite angular point of the parallelepiped.

Then, if ρ be the density, and u the velocity parallel to x, at the point P, the quantity of fluid which enters the parallelepiped at the face $\beta\gamma$, containing P, will be

$$\rho u\beta\gamma\delta t,$$

in the time δt, and therefore the quantity which, during the same time, flows out at the opposite face, will be

$$\left(\rho u + \frac{d \cdot \rho u}{dx}\alpha\right)\beta\gamma\delta t.$$

Hence the loss of fluid in consequence of the motion parallel to x, is

$$\frac{d(\rho u)}{dx}\alpha\beta\gamma\delta t.$$

Similarly the quantities lost in consequence of the other motions, are

$$\frac{d(\rho v)}{dy}\alpha\beta\gamma\delta t, \text{ and } \frac{d(\rho w)}{dz}\alpha\beta\gamma\delta t,$$

and the total loss is,

$$\left\{\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right\}\alpha\beta\gamma\delta t;$$

but the increase in the quantity of fluid in the time δt is given by the expression $\frac{d\rho}{dt}\delta t.\alpha\beta\gamma$, that is, the loss is

$$-\frac{d\rho}{dt}\alpha\beta\gamma\delta t,$$

and therefore, equating these expressions,

$$\frac{d\rho}{dt}+\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}=0.$$

If the fluid be homogeneous and incompressible, ρ is constant, and the equation becomes

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

This last equation is also true, if the fluid be heterogeneous, provided it be incompressible; for the density of a particle in motion will be invariable, and therefore the variation of ρ, considered as a function of x, y, z, and t, will be zero, if we take

$$\delta x = u\delta t, \quad \delta y = v\delta t, \quad \text{and } \delta z = w\delta t.$$

Hence

$$\frac{d\rho}{dt}+u\frac{d\rho}{dx}+v\frac{d\rho}{dy}+w\frac{d\rho}{dz}=0,$$

and, subtracting this from the general equation of continuity, we get

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

134. We can also form the equation of continuity by following the motion of a small element of fluid, and by expressing the fact that its mass remains unchanged during the element of time δt.

If x, y, z be the co-ordinates of a particle at the time t, its co-ordinates at the time $t + \delta t$ are $x + u\delta t$, $y + v\delta t$, $z + w\delta t$.

Taking an element $\rho\delta x\delta y\delta z$, ρ becomes $\rho + \frac{D\rho}{dt}\delta t$,

and the new values of δx, δy, δz are

$$\delta x + \frac{du}{dx}\delta x\delta t,\ \delta y + \frac{dv}{dy}\delta y\delta t,\ \delta z + \frac{dw}{dz}\delta z\delta t;$$

multiplying these together, and equating the product to $\rho\delta x\delta y\delta z$, we obtain

$$\frac{D\rho}{dt} + \rho\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0,$$

$$\text{or}\quad \frac{1}{\rho}\frac{D\rho}{dt} + \theta = 0,$$

where θ is the rate of dilatation.

As before, the equation may be written in the form

$$\frac{d\rho}{dt} + \frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0.$$

135. The *Integral* Equation of Continuity is another expression of the same condition. (Lagrange, *Méc. Analytique.*)

Let a, b, c be the co-ordinates of a particle P at any given epoch,

x, y, z of the same particle at the time t.

Take $PABC$, a small tetrahedron in the fluid having its edges PA, PB, PC parallel to the co-ordinate axes.

At the time t the element of fluid, occupying the space $PABC$ at the given epoch, will form a differently situated tetrahedron $P'A'B'C'$, and, x, y, z being the co-ordinates of P', the co-ordinates of A', relative to P', will be

$$\frac{dx}{da}\delta a,\quad \frac{dy}{da}\delta a,\quad \frac{dz}{da}\delta a,$$

of B', $\frac{dx}{db}\delta b,\quad \frac{dy}{db}\delta b,\quad \frac{dz}{db}\delta b,$

of C', $\frac{dx}{dc}\delta c,\quad \frac{dy}{dc}\delta c,\quad \frac{dz}{dc}\delta c.$

The volume of the tetrahedron $P'A'B'C'$ therefore

$$=\frac{1}{6}\begin{vmatrix} \frac{dx}{da}\delta a, & \frac{dy}{da}\delta a, & \frac{dz}{da}\delta a \\ \frac{dx}{db}\delta b, & \frac{dy}{db}\delta b, & \frac{dz}{db}\delta b \\ \frac{dx}{dc}\delta c, & \frac{dy}{dc}\delta c, & \frac{dz}{dc}\delta c \end{vmatrix}$$

$$=\frac{1}{6}\,\delta a\delta b\delta c\,\frac{d(x, y, z)}{d(a, b, c)}=\frac{1}{6}J.\,\delta a\delta b\delta c,$$

employing the usual notation for the Jacobian; also the mass of the element $=\frac{1}{6}\rho J\delta a\delta b\delta c$;

$$\therefore\ \frac{d}{dt}(\rho J)=0.$$

If ρ_0 be the initial density, the expression for the mass is initially $\frac{1}{6}\rho_0\,\delta a\delta b\delta c$,

$$\text{and } \therefore\ \rho J=\rho_0.$$

136. *Particular cases of the equation of continuity.* The equation of continuity can in some cases be integrated, as in the two following.

(1) *A homogeneous liquid moves in one plane, the motions of all its particles being symmetrical with regard to a fixed centre.*

Taking r as the distance from the centre the velocity (V) is a function of r, and

$$u=V\frac{x}{r},\quad v=V\frac{y}{r},\quad w=0.$$

The equation of continuity is $\frac{du}{dx}+\frac{dv}{dy}=0$,

and, by transformation, it becomes

$$\frac{dV}{dr}+\frac{V}{r}=0, \text{ or } \frac{d}{dr}(rV)=0.$$

Hence we obtain

$$rV=f(t),$$

and therefore, at any given time, the velocity is inversely proportional to the distance.

(2) *The motion of a homogeneous liquid is symmetrical in all directions, with regard to a fixed centre.*

In this case, $u=V\frac{x}{r}$, $v=V\frac{y}{r}$, $w=V\frac{z}{r}$,

and, by transformation, the equation of continuity,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

becomes $$\frac{dV}{dr}+\frac{2V}{r}=0, \text{ or } \frac{d}{dr}(r^2V)=0.$$

Hence $$r^2V=f(t).$$

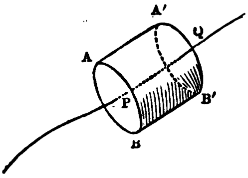
These two results can however be obtained without going through the process of integration.

For, in the second case, it is an obvious condition of the continuity of the motion, that the quantity of liquid, which during a small interval of time, at a given epoch, flows across a spherical surface, is the same whatever be the radius, and therefore $4\pi r^2V$ is independent of r, and is a function of the time only.

In a similar manner the result for the first case, $rV=f(t)$, can be obtained.

137. Another form of the equation of continuity may also be given.

Let $PQ=\delta s$ be an arc of the line of motion passing through a point Q; and let AB be a small area normal to the arc, such that all the particles of fluid crossing it may be considered as moving perpendicular to it.



Let AA', BB', &c. be small arcs of the lines of motion through the bounding points of AB, and $A'B'$ the normal section through Q of the surface formed by these lines of motion.

Take ρ as the density of the fluid in PQ at the time t, κ the area of AB, and v the velocity at P; then the quantity of fluid which enters at AB during the time δt

$$= \kappa\rho v\delta t,$$

and that which flows out at $A'B'$

$$= \kappa\rho v\delta t + \frac{d}{ds}(\kappa\rho v\delta t) \,.\, \delta s.$$

The excess of the former over the latter of these two expressions is the whole increase of the fluid in PQ during the time δt, and is

$$-\frac{d}{ds}(\kappa\rho v)\,\delta t\delta s:$$

but the mass of fluid at the time t being $\kappa\rho\delta s$, the increase in the time δt is also expressed by

$$\frac{d}{dt}(\kappa\rho\delta s)\,\delta t, \quad \text{or} \quad \frac{d}{dt}(\kappa\rho)\,\delta s\delta t,$$

and therefore

$$\frac{d}{dt}(\kappa\rho) + \frac{d}{ds}(\kappa\rho v) = 0.$$

From the way in which this equation has been obtained, it will be seen that allowance is made for the expansion of the element which may in certain cases take place, and it is only in this way that κ can be an explicit function of the time. The small section AB may be taken arbitrarily, but the section $A'B'$ will depend, not only on the arc PQ, but also on the directions of the lines of motion passing through the bounding curve of AB; the variation of κ may therefore depend on the time explicitly, since these lines of motion may vary with the time.

138. *The Bounding Surface.* Regarding a fluid as a continuous mass, we shall assume that particles on the surface remain on the surface.

Hence if (x, y, z) be co-ordinates of a particle at the time t, and if

$$F(x, y, z, t) = 0$$

be the equation to the surface; then, at the time $t + \delta t$,

$$F(x + u\delta t,\ y + v\delta t,\ z + w\delta t,\ t + \delta t) = 0,$$

and therefore $$\frac{dF}{dt} + u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz} = 0,$$

the differential equation of the surface.

This equation is equally true for any surface inside the fluid, composed of a given sheet of particles of fluid, and, of a closed surface, enclosing always the same fluid.

It is also the analytical expression of the fact that the velocity of the fluid at the surface perpendicular to the surface is equal to the normal velocity of the surface.

If l, m, n be the direction-cosines of the normal at any point x, y, z of the bounding surface, and H the normal velocity outwards,

$$H = lu + mv + nw$$

$$= \frac{uF'x + vF'y + wF'z}{R}$$

where $R^2 = (F'x)^2 + (F'y)^2 + (F'z)^2$;

$$\therefore\ H = -\frac{F'(t)}{R}.$$

139. *To find the equations of motion.*

At the time t, let u, v, w, be the velocities, parallel to the axes, of the fluid at the point x, y, z: u, v, w are therefore functions of x, y, z, and t, which are the independent variables.

Let m be the mass of an element of fluid about the point x, y, z, and let mX, mY, mZ, be the impressed forces acting upon m.

The effective forces are

$$m\frac{Du}{dt},\quad m\frac{Dv}{dt},\quad m\frac{Dw}{dt}$$

and, by D'Alembert's principle, the aggregate of these forces reversed, would, in combination with the impressed forces, maintain the equilibrium of the fluid.

Hence, if p be the pressure, we obtain from Art. (14) the equations

$$\left.\begin{aligned}\frac{dp}{dx} &= \rho\left(X - \frac{Du}{dt}\right),\\ \frac{dp}{dy} &= \rho\left(Y - \frac{Dv}{dt}\right),\\ \frac{dp}{dz} &= \rho\left(Z - \frac{Dw}{dt}\right),\end{aligned}\right\} \quad \text{...............} (1),$$

where $$\frac{Du}{dt} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz},$$

with similar equations for $\frac{Dv}{dt}$ and $\frac{Dw}{dt}$.

Substituting, we obtain, as the equations of motion,

$$\left.\begin{aligned}\frac{1}{\rho}\frac{dp}{dx} &= X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz},\\ \frac{1}{\rho}\frac{dp}{dy} &= Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz},\\ \frac{1}{\rho}\frac{dp}{dz} &= Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}.\end{aligned}\right\} \quad \text{.........} (2).$$

If the fluid be elastic, and, if we suppose the temperature constant, we have also the equation,

$$p = \kappa\rho.$$

In the case of a liquid, if Π be the external pressure upon its surface, and p the pressure of the liquid at the surface, we shall have

$$p = \Pi,$$

and therefore, at all points of the free surface,

$$\frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz} = \frac{d\Pi}{dt}.$$

140. If the forces be such that $Xdx + Ydy + Zdz = -dV$, and if the motion be of such a nature that $udx + vdy + wdz = d\phi$, that is, if the forces are derivable from a potential, and the velocities from a velocity-function, the equations can be reduced to a simpler form.

In this case $\dfrac{d\phi}{dx} = u$, $\dfrac{d\phi}{dy} = v$, $\dfrac{d\phi}{dz} = w$;

$$\text{and } \therefore\ \frac{Du}{dt} = \frac{d^2\phi}{dxdt} + \frac{d\phi}{dx}\cdot\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\cdot\frac{d^2\phi}{dxdy} + \frac{d\phi}{dz}\cdot\frac{d^2\phi}{dxdz},$$

$$\frac{Dv}{dt} = \frac{d^2\phi}{dydt} + \frac{d\phi}{dx}\cdot\frac{d^2\phi}{dxdy} + \frac{d\phi}{dy}\cdot\frac{d^2\phi}{dy^2} + \frac{d\phi}{dz}\cdot\frac{d^2\phi}{dydz},$$

$$\frac{Dw}{dt} = \frac{d^2\phi}{dzdt} + \frac{d\phi}{dx}\cdot\frac{d^2\phi}{dxdz} + \frac{d\phi}{dy}\cdot\frac{d^2\phi}{dydz} + \frac{d\phi}{dz}\cdot\frac{d^2\phi}{dz^2}.$$

From the equations (1) we have

$$\frac{1}{\rho}\,dp = \left(X - \frac{Du}{dt}\right)dx + \left(Y - \frac{Dv}{dt}\right)dy + \left(Z - \frac{Dw}{dt}\right)dz;$$

$$\text{and } \therefore\ \frac{1}{\rho}\,dp = -dV - d\,.\,\frac{d\phi}{dt} - \frac{1}{2}d\,.\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\},$$

$$\text{or}\qquad \frac{1}{\rho}\,dp = -dV - d\,.\,\frac{d\phi}{dt} - \frac{1}{2}d\,(q^2)\ \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

q being the resultant velocity at the point x, y, z.

Hence, if the fluid be inelastic and homogeneous,

$$\frac{p}{\rho} + V + \frac{1}{2}q^2 + \frac{d\phi}{dt} = C,$$

and the equation of continuity is

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

If the fluid be elastic, $p = k\rho$, and we obtain

$$k\log p + V + \frac{d\phi}{dt} + \frac{1}{2}q^2 = C,$$

$$\text{or, generally,}\qquad \int\frac{dp}{\rho} + V + \frac{d\phi}{dt} + \frac{1}{2}q^2 = C.$$

In each integration an arbitrary function of t must be intro-

duced, but it is unnecessary to insert such a function in the equation, as it may be supposed to be contained in $\dfrac{d\phi}{dt}$.

141. Taking s an arc of the line of motion passing through the point x, y, z, then $-m\dfrac{dV}{ds}$ is the force on the particle m in the direction of its motion, and the velocity

$$q = u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds} = \frac{d\phi}{ds};$$

and therefore, if mS be the tangential force on m, the equation (5) may be written

$$\frac{1}{\rho}\frac{dp}{ds} = S - \frac{dq}{dt} - q\frac{dq}{ds}.$$

This equation may be obtained more briefly as follows. Taking p as the pressure at any point of a fluid at rest, and measuring s in any direction, we have

$$\frac{dp}{ds} = \rho S,$$

where mS measures the force on m in the direction of s.

In the case of the fluid in motion, measure s in the direction of the line of motion, and let q be the velocity; then q is a function of s and t, and

$$\frac{Dq}{dt} = \frac{dq}{dt} + q\frac{dq}{ds}.$$

Hence, by D'Alembert's principle,

$$\frac{1}{\rho}\frac{dp}{ds} = S - \frac{dq}{dt} - q\frac{dq}{ds} \quad \text{.................. (6).}$$

142. Cases of motion are of course conceivable in which $udx + vdy + wdz$ is not a complete differential, and in such cases we must employ the equations (1) in order to determine the pressure at any point.

For instance, if a mass of liquid revolve uniformly, without change of form or relative displacement, about a fixed axis, there is no velocity-function.

Thus, taking the fixed axis as the axis of z,

$$u = -\omega y, \quad v = \omega x, \quad w = 0,$$

$$\therefore\ udx + vdy + wdz = \omega(xdy - ydx),$$

an expression which is not an exact differential.

In this case, recurring to the fundamental equations of motion, we have

$$\frac{1}{\rho}\frac{dp}{dx} = X + \omega^2 x, \quad \frac{1}{\rho}\frac{dp}{dy} = Y + \omega^2 y, \quad \frac{1}{\rho}\frac{dp}{dz} = Z,$$

and therefore,

$$\frac{1}{\rho}dp = Xdx + Ydy + Zdz + \omega^2(xdx + ydy),$$

as in Art. (31).

143. The equations of motion, obtained in Article (139) by the force method, are Euler's Equations.

We now proceed to establish the same equations by the Flux method, as suggested by Mr. A. G. Greenhill.

In this method we consider the changes of momentum which take place within an element of space $\alpha\beta\gamma$, and which are due to the action of the fluid pressures on the surfaces of the element.

In other words, we consider the action of stationary force upon variable matter.

The component, parallel to x, of the momentum of the fluid within the space is $\rho u\alpha\beta\gamma$ at the time t, and, at the time $t + \delta t$, it is

$$\rho u\alpha\beta\gamma + \frac{d(\rho u)}{dt}\delta t\alpha\beta\gamma.$$

During the time δt the momentum which enters at the face x is $\rho u^2\delta t\beta\gamma$, and the momentum which emerges at the face $x + \alpha$ is

$$\rho u^2\delta t\beta\gamma + \frac{d(\rho u^2)}{dx}\delta t\alpha\beta\gamma;$$

the momentum, parallel to x, which enters at the face y is $\rho uv\delta t\gamma\alpha$, and that which emerges at the face $y + \beta$ is

$$\rho uv\delta t\gamma\alpha + \frac{d(\rho uv)}{dy}\delta t\alpha\beta\gamma;$$

and similarly the loss of momentum at the faces z and $z+\gamma$ is

$$\frac{d(\rho uw)}{dz}\delta t\alpha\beta\gamma.$$

Considering the pressures which act on the element, the pressure p on the face x generates in the time δt the momentum $p\delta t\beta\gamma$, and the pressure $p+\frac{dp}{dx}\delta x$ on the face $x+\alpha$ generates the momentum

$$-p\delta t\beta\gamma - \frac{dp}{dx}\delta t\alpha\beta\gamma.$$

Again, the impressed force X generates the momentum

$$\rho X\delta t\alpha\beta\gamma,$$

and this impressed force, together with the pressures, produce the momentum generated in the space $\alpha\beta\gamma$.

Hence, equating the momenta due to the force and pressures, to the momentum actually generated within the space, and dividing by $\delta t\alpha\beta\gamma$, we obtain

$$\rho X - \frac{dp}{dx} = \frac{d(\rho u)}{dt} + \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz},$$

with similar equations for y and z.

Taking account of the equation of continuity these equations are at once reduced to Euler's Equations.

144. *Lagrange's Equations.*

In the form given by Lagrange of the equations of motion the time and the initial co-ordinates of a particle are the independent variables.

If a, b, c be the initial co-ordinates of a particle, and x, y, z the co-ordinates of the same particle at the time t, x, y, z, u, v, and w are functions of t, a, b, and c, which are the independent variables; and the component accelerations of the particle of fluid originally at (a, b, c) are, at the time t,

$$\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}.$$

Assuming the existence of a force potential V, and putting dP for $\frac{dp}{\rho}$, the equations of motion, by D'Alembert's principle are

$$\frac{dP}{dx} = -\frac{dV}{dx} - \frac{du}{dt},$$

$$\frac{dP}{dy} = -\frac{dV}{dy} - \frac{dv}{dt},$$

$$\frac{dP}{dz} = -\frac{dV}{dz} - \frac{dw}{dt};$$

or, if $P + V = Q$, the equations are

$$\frac{du}{dt} = -\frac{dQ}{dx}, \quad \frac{dv}{dt} = -\frac{dQ}{dy}, \quad \frac{dw}{dt} = -\frac{dQ}{dz}.$$

Multiplying by $\frac{dx}{da}, \frac{dy}{da}, \frac{dz}{da}$, and adding, we obtain

$$-\frac{dQ}{da} = \frac{du}{dt}\frac{dx}{da} + \frac{dv}{dt}\frac{dy}{da} + \frac{dw}{dt}\frac{dz}{da} \quad \ldots\ldots\ldots\ldots\ldots (\alpha),$$

and, similarly,

$$-\frac{dQ}{db} = \frac{du}{dt}\frac{dx}{db} + \frac{dv}{dt}\frac{dy}{db} + \frac{dw}{dt}\frac{dz}{db} \quad \ldots\ldots\ldots\ldots\ldots (\beta),$$

$$-\frac{dQ}{dc} = \frac{du}{dt}\frac{dx}{dc} + \frac{dv}{dt}\frac{dy}{dc} + \frac{dw}{dt}\frac{dz}{dc} \quad \ldots\ldots\ldots\ldots\ldots (\gamma).$$

These equations, together with the integral equation of continuity,

$$\frac{d}{dt}(\rho J) = 0,$$

are Lagrange's Hydrodynamical Equations.

145. *Cauchy's Integrals.*

Eliminating Q by differentiation from the equations (β) and (γ), we obtain

$$\frac{d^2u}{dtdb}\frac{dx}{dc} - \frac{d^2u}{dtdc}\frac{dx}{db} + \frac{d^2v}{dtdb}\frac{dy}{dc} - \frac{d^2v}{dtdc}\frac{dy}{db} + \frac{d^2w}{dtdb}\frac{dz}{dc} - \frac{d^2w}{dtdc}\frac{dz}{db} = 0.$$

Integrate this expression with regard to t, and take u_0, v_0, w_0 as initial values; then

$$\frac{du}{db}\frac{dx}{dc}-\frac{du}{dc}\frac{dx}{db}+\frac{dv}{db}\frac{dy}{dc}-\frac{dv}{dc}\frac{dy}{db}+\frac{dw}{db}\frac{dz}{dc}-\frac{dw}{dc}\frac{dz}{db}=\frac{dw_0}{db}-\frac{dv_0}{dc}.$$

Now $$\frac{du}{da}=\frac{du}{dx}\frac{dx}{da}+\frac{du}{dy}\frac{dy}{da}+\frac{du}{dz}\frac{dz}{da},\ \&c.,$$

and, making these substitutions, the equation becomes

$$\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\frac{d(y,z)}{d(b,c)}+\left(\frac{du}{dz}-\frac{dw}{dx}\right)\frac{d(z,x)}{d(b,c)}+\left(\frac{dv}{dx}-\frac{du}{dy}\right)\frac{d(x,y)}{d(b,c)}=\frac{dw_0}{db}-\frac{dv_0}{dc}.$$

Assuming that

$$\frac{dw}{dy}-\frac{dv}{dz}=2\xi,\quad \frac{du}{dz}-\frac{dw}{dx}=2\eta,\quad \frac{dv}{dx}-\frac{du}{dy}=2\zeta,$$

we obtain the equations,

$$\xi\frac{d(y,z)}{d(b,c)}+\eta\frac{d(z,x)}{d(b,c)}+\zeta\frac{d(x,y)}{d(b,c)}=\xi_0,$$

$$\xi\frac{d(y,z)}{d(c,a)}+\eta\frac{d(z,x)}{d(c,a)}+\zeta\frac{d(x,y)}{d(c,a)}=\eta_0,$$

$$\xi\frac{d(y,z)}{d(a,b)}+\eta\frac{d(z,x)}{d(a,b)}-\zeta\frac{d(x,y)}{d(a,b)}=\zeta_0.$$

Multiplying by $\frac{dx}{da}$, $\frac{dx}{db}$, $\frac{dx}{dc}$, and adding, we obtain

$$J\xi=\xi_0\frac{dx}{da}+\eta_0\frac{dx}{db}+\zeta_0\frac{dx}{dc},$$

$$J\eta=\xi_0\frac{dy}{da}+\eta_0\frac{dy}{db}+\zeta_0\frac{dy}{dc},$$

$$J\zeta=\xi_0\frac{dz}{da}+\eta_0\frac{dz}{db}+\zeta_0\frac{dz}{dc},$$

where $$J=\frac{d(x,y,z)}{d(a,b,c)}.$$

Take a rectangular parallelepiped $\delta a\delta b\delta c$, the co-ordinates of the ends of a diagonal being a, b, c and $a+\delta a$, $b+\delta b$, $c+\delta c$.

At the time t this element of fluid will be in the shape of a small space bounded by six curved surfaces, forming ultimately an oblique parallelepiped, the volume of which is*

$$\frac{d(x, y, z)}{d(a, b, c)} \delta a \delta b \delta c.$$

Hence, if ρ_0 be the initial density,

$$J\rho = \rho_0,$$

and therefore

$$\frac{\xi}{\rho} = \frac{\xi_0}{\rho_0}\frac{dx}{da} + \frac{\eta_0}{\rho_0}\frac{dx}{db} + \frac{\zeta_0}{\rho_0}\frac{dx}{dc},$$

$$\frac{\eta}{\rho} = \frac{\xi_0}{\rho_0}\frac{dy}{da} + \frac{\eta_0}{\rho_0}\frac{dy}{db} + \frac{\zeta_0}{\rho_0}\frac{dy}{dc},$$

$$\frac{\zeta}{\rho} = \frac{\xi_0}{\rho_0}\frac{dz}{da} + \frac{\eta_0}{\rho_0}\frac{dz}{db} + \frac{\zeta_0}{\rho_0}\frac{dz}{dc}.$$

If a velocity-function exist, that is, if the component velocities are the differential co-efficients of a function ϕ, so that

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz},$$

the expressions ξ, η, ζ all vanish.

From the preceding equations it follows that these quantities are always zero if their initial values vanish.

When a velocity-function exists the motion is said to be irrotational, and we have therefore the theorem that *the motion of a fluid under the action of natural forces, if once irrotational, is always irrotational.*

When a velocity-function does not exist, the motion is said to be rotational.

The reason for the phraseology employed to distinguish between the two kinds of motion is given in the following article, which is taken from a paper, by Professor Stokes, in the eighth volume of the *Cambridge Philosophical Transactions.*

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* See Todhunter's *Integral Calculus*, Art. 247.

146. *Physical Interpretation.*

Conceive an indefinitely small element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed; let the form of the element be so taken that the resulting solid shall be that which is the simplest with respect to rotatory motion, namely, that which has its three principal moments about axes passing through the centre of gravity equal to each other, and therefore every axis passing through that point a principal axis, and consider the linear and angular motions of the element immediately after solidification.

By the instantaneous solidification velocities will be suddenly generated or destroyed in the different portions of the element, and a set of impulsive forces will be called into action. Let x, y, z be the co-ordinates of the centre of gravity G of the element at the instant of solidification, $x+x'$, $y+y'$, $z+z'$ those of any other point in it.

Let u, v, w be the velocities of G along the three axes just before solidification, u', v', w' the relative velocities of the point whose relative co-ordinates are x', y', z'.

Let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the velocities of G, $u_{\prime}$, $v_{\prime}$, $w_{\prime}$ the relative velocities of the point $(x'y'z')$, and ξ, η, ζ the angular velocities just after solidification.

Since all the impulsive forces are internal,

$$\bar{u}=u, \quad \bar{v}=v, \quad \bar{w}=w.$$

Also, by the conservation of angular momentum,

$$\Sigma m\,\{y'(w_{\prime}-w')-z'(v_{\prime}-v')\}=0, \text{ \&c.}$$

m denoting an element of the mass considered.

But $$u_{\prime}=\eta z'-\zeta y',$$

$$u'=\frac{du}{dx}x'+\frac{du}{dy}y'+\frac{du}{dz}z', \text{ ultimately,}$$

and similar expressions hold good for the other quantities.

Substituting in the above equations, and observing that

$$\Sigma\,(my'z')=0, \quad \Sigma\,(mz'x')=0, \quad \Sigma\,(mx'y')=0, \text{ and}$$

$\Sigma(mx'^2) = \Sigma(my'^2) = \Sigma(mz'^2)$, we have

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz}, \quad 2\eta = \frac{du}{dz} - \frac{dw}{dx}, \quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy}.$$

We see then that an indefinitely small element of the fluid of which the three principal moments about the centre o gravity are equal, if suddenly solidified and detached from th rest of the fluid, will begin to move with a motion simply o translation, which may however vanish, or a motion of transla tion combined with one of rotation, according as $udx + vdy + wd$ is or is not an exact differential.

147. Recurring to Euler's equations, and assuming th existence of a Force Potential, we have

$$-\frac{dQ}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz},$$

$$-\frac{dQ}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz},$$

$$-\frac{dQ}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}.$$

Differentiating the third of these equations with respect y, the second with respect to z, and subtracting, we obtain

$$\frac{D}{dt}\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + \frac{du}{dy}\frac{dw}{dx} + \frac{dv}{dy}\frac{dw}{dy} + \frac{dw}{dy}\cdot\frac{dw}{dz}$$
$$- \frac{du}{dz}\frac{dv}{dx} - \frac{dv}{dz}\frac{dv}{dy} - \frac{dw}{dz}\cdot\frac{dv}{dz}$$

If we add and subtract $\frac{dv}{dx}\cdot\frac{dw}{dx}$, the above equation

written

$$\frac{D\xi}{dt} = \frac{dv}{dx}\cdot\eta + \frac{dw}{dx}\cdot\zeta - \left(\frac{dv}{dy} + \frac{dw}{dz}\right)\xi.$$

In the case of a homogeneous liquid, the equatio

tinuity gives $$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

and we then have the equations

$$\frac{D\xi}{dt}=\xi\frac{du}{dx}+\eta\frac{dv}{dx}+\zeta\frac{dw}{dx},$$
$$\frac{D\eta}{dt}=\xi\frac{du}{dy}+\eta\frac{dv}{dy}+\zeta\frac{dw}{dy},$$
$$\frac{D\zeta}{dt}=\xi\frac{du}{dz}+\eta\frac{dv}{dz}+\zeta\frac{dw}{dz}.$$

Also, observing that

$$\eta\frac{dv}{dx}+\zeta\frac{dw}{dx}=\eta\left(2\zeta+\frac{du}{dy}\right)+\zeta\left(\frac{du}{dz}-2\eta\right)=\eta\frac{du}{dy}+\zeta\frac{du}{dz},$$

the equations take the forms

$$\frac{D\xi}{dt}=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz},$$
$$\frac{D\eta}{dt}=\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz},$$
$$\frac{D\zeta}{dt}=\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\frac{dw}{dz}.$$

These equations are given by Professor Stokes in the paper before referred to, and also in a paper by Professor Helmholtz in the volume for the year 1858 of Crelle's Journal*.

148. *Impulsive action.*

If impulsive forces be made to act in a liquid, or if impulsive pressures be excited by a sudden change of motion in a mass of liquid, it can be shewn, exactly as in Articles (7) and (9), that the impulsive pressure at any point is the same in every direction, and that any impulsive pressure is transmitted equally through the liquid.

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* A translation of this paper by Professor Tait is given in the supplementary number for July, 1867, of the *Philosophical Magazine.*

Professor Nanson, in the *Mathematical Messenger* for 1873, extends the equations to the case of any fluid.

Thus, for any fluid, $\dfrac{D\xi}{dt}=\xi\dfrac{du}{dx}+\eta\dfrac{dv}{dx}+\zeta\dfrac{dw}{dx}-\xi\theta,$

where $\theta=\dfrac{du}{dx}+\dfrac{dv}{dy}+\dfrac{dw}{dz},$

and the equation of continuity is $\dfrac{1}{\rho}\dfrac{D\rho}{dt}+\theta=0.$

Eliminating θ, we obtain $\dfrac{D}{dt}\left(\dfrac{\xi}{\rho}\right)=\dfrac{\xi}{\rho}\dfrac{du}{dx}+\dfrac{\eta}{\rho}\dfrac{dv}{dx}+\dfrac{\zeta}{\rho}\dfrac{dw}{dx}.$

We may suppose, for instance, a closed vessel full of liquid, and an impulsive pressure P applied to it by means of a piston, area K, fitting in the side of the vessel; the impulsive pressure at all points will be the same, and will be measured by the quantity $\frac{P}{K}$.

To find the relation between the impulsive pressure and the change of velocity.

Imagine impulsive action transmitted in any way through the liquid. Let u, v, w be the velocities at any point P, (x, y, z) immediately before the impulse, and u', v', w' immediately after, and let ϖ be the impulsive action at P.

Suppose a small prism PQ, having its axis parallel to x, to be solidified; then, since the impulse at $Q = \varpi + \frac{d\varpi}{dx}\delta x$,

$$-\kappa\frac{d\varpi}{dx}\delta x = \kappa\rho\delta x\,(u' - u), \text{ where } \kappa \text{ is the sectional area,}$$

$$\text{or} \quad \frac{d\varpi}{dx} + \rho\,(u' - u) = 0.$$

Similarly,
$$\frac{d\varpi}{dy} + \rho\,(v' - v) = 0,$$
$$\frac{d\varpi}{dz} + \rho\,(w' - w) = 0;$$

$$\therefore \frac{d\varpi}{\rho} + (u' - u)\,dx + (v' - v)\,dy + (w' - w)\,dz = 0.$$

149. The following problems will serve to illustrate the application to particular cases of the principles of Hydrodynamics.

(1) *A vessel containing liquid moves vertically upwards with an uniform acceleration; required to find the pressure at any point.*

Let f be the upward acceleration, and therefore mf the effective force on a particle m of the fluid.

Measuring z downwards, and reversing the effective forces,

$$dp = \rho\,(g + f)\,dz,$$

and
$$p = C + \rho\,(g + f)\,z.$$

Let the pressure at the free surface be supposed constant and be represented by Π; then, if z', and $z'+x$, be the vertical distances from the origin of the free surface and of any other horizontal plane in the fluid,

$$\Pi = C + \rho (g+f) z',$$

$$p = C + \rho (g+f)(z'+x),$$

and therefore $$p = \rho (g+f) x + \Pi.$$

This result might also have been obtained by arguing that the resultant fluid pressure on any portion, elementary or finite, of the fluid, produces, in combination with the force of gravity, an upward acceleration f, and therefore that forces mf, acting vertically downwards, would produce the same pressure at any point of the fluid, supposed at rest. By such reasoning the problem is at once, apparently without the intervention of D'Alembert's principle, placed in the domain of Hydrostatics, and, taking axes fixed relatively to the fluid, the equilibrium equation becomes applicable, and leads to the value of p just obtained.

The reasoning of Art. (30), in which the equilibrium of a revolving fluid is discussed, is of the same kind, and it must be noticed that in each case an assumption is made, which is really equivalent to the application of D'Alembert's principle, although for these cases the enunciation of the principle in its most general form is unnecessary. It is in fact assumed implicitly, that, when there is no relative displacement of the fluid particles, the molecular actions are the same as if the fluid were at rest in the same form, or that, if it be conceived possible that the motion would call into play additional molecular actions, no alteration is produced in the pressure by such actions, and the pressure consequently depends on the force of gravity and the hypothetical forces mf in the present case, and, in the case of Art. (29), on the hypothetical forces $m\omega^2 r$ in combination with the force of gravity.

(2) *An open vessel containing liquid is suddenly moved upwards with a given velocity.*

Measuring z downwards from the surface of the liquid,

$$d\varpi - \rho w' dz = 0,$$

$$\therefore \ \varpi = \rho w' z,$$

since there is no impulse at the surface.

In this case the theorems of Arts. 32, 39, and 40 are equally true for impulsive pressures.

(3) *A closed vessel, the interior surface of which is spherical, is filled with heavy inelastic fluid, and the vessel is moved in any way; it is required to find, at any instant, the surfaces of equal pressure.*

Supposing the surface smooth and the fluid initially at rest, it is clear that no rotation can be caused in the fluid, and therefore that the actual motion of every particle of the fluid will be the same as that of the centre of the sphere. At any given instant, let f be the acceleration, in a known direction, of the centre of the sphere; then, by D'Alembert's principle, the fluid may be supposed at rest under the action of gravity and the reversed forces mf, and, since the resultant of the acceleration g, and the reversed acceleration f, is the same, both in magnitude and direction, for all particles of the fluid, it follows that the surfaces of equal pressure are, at the given instant, planes perpendicular to the direction of that resultant.

(4) *A quantity of liquid moves in a straight tube of small bore under the action of a force to a point in the tube, which is proportional to the distance from that point. It is required to determine the motion and the pressure.*

Let $2l$ be the length of tube occupied by the liquid, and z the distance of the nearer free surface from the centre of force (O). Then if p be the pressure and u the velocity at a distance x from O,

$$\frac{dp}{\rho} = \left(-\mu x - \frac{du}{dt}\right) dx,$$

$$\text{and } \frac{p}{\rho} = f(t) - \mu \frac{x^2}{2} - x \frac{du}{dt}.$$

The pressure vanishes when $x = z$ and when $x = z + 2l$,

$$\therefore \frac{du}{dt} = -\mu(z+l),$$

$$\text{or} \quad \frac{d^2z}{dt^2} = -\mu(z+l);$$

$$\text{and} \therefore z + l = A\cos(\sqrt{\mu}\,t + \alpha),$$

the constants being determined by the initial position.

Also
$$\frac{p}{\rho} = -\mu\frac{x^2 - z^2}{2} - (x-z)\frac{du}{dt}$$

$$= -\mu\frac{x^2 - z^2}{2} + \mu(x-z)(z+l),$$

and the pressure at any distance x is determined.

(5) *A vertical tube* AB *of small section has two apertures close to its base* B *in which horizontal tubes are fitted, and the apertures are closed by valves; a given height* (*a*) *of the tube* AB *is filled with water and the valves are then opened. The areal section of each horizontal tube being half that of the vertical tube, and the length of each greater than* AB, *it is required to determine the motion.*

Let z be the height above B of the free surface in AB at the time t,

z' the distance from B of the free surface in each lower tube; then κ being the areal section of AB, the volume of liquid is $\kappa z + 2.\frac{\kappa}{2}.z'$;

$$\therefore z + z' = a.$$

If p be the pressure and u the velocity, measured upwards, at a height x in AB,

$$\frac{dp}{\rho} = \left(-g - \frac{du}{dt}\right)dx,$$

$$\frac{p}{\rho} = f(t) - gx - x\frac{du}{dt};$$

and p vanishes when $x = z$;

$$\therefore \frac{p}{\rho} = g(z-x) + (z-x)\frac{du}{dt},$$

and if $p=p'$ at B,

$$\frac{p'}{\rho}=gz+z\frac{du}{dt}.$$

Similarly, if u' be the velocity in a lower tube

$$\frac{p'}{\rho}=z'\frac{du'}{dt},$$

$$\text{and } \therefore\ z'\frac{du'}{dt}=gz+z\frac{du}{dt}.$$

$$\text{But } u=\frac{dz}{dt}, \text{ and } u'=\frac{dz'}{dt}=-\frac{dz}{dt};$$

$$\therefore\ a\frac{d^2z}{dt^2}+gz=0,$$

$$\text{and } z=A\cos\left(\sqrt{\frac{g}{a}}\,t+\alpha\right)=a\cos\sqrt{\frac{g}{a}}\,t.$$

Hence the water will flow out of the vertical tube in the time $\frac{\pi}{2}\sqrt{\frac{a}{g}}$.

(6) *Two rigid laminæ, in one of which is a very small circular aperture, are placed very near to each other with their planes parallel. Supposing air to be rushing through the aperture, it is required to form the differential equation of motion.*

It being supposed that no forces are in action, and that the motion is symmetrical with regard to the aperture, the equation for the density at a distance r is

$$\frac{k}{\rho}\frac{d\rho}{dr}=-\frac{dv}{dt}-v\frac{dv}{dr}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

since $p=k\rho$; and the equation of continuity is

$$\frac{d\rho}{dt}+v\frac{d\rho}{dr}+\rho\left(\frac{dv}{dr}+\frac{v}{r}\right)=0\,*.$$

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* This can be obtained, by transformation, from the equation of Art. (133), or from Art. (137). It is however at once deducible from consideration of the quantities of fluid, which flow, during the time δt, across the circles of radii r and $r+\delta r$.

Eliminating $\frac{d\rho}{dr}$, we get

$$\frac{k}{\rho}\frac{d\rho}{dt} = k\frac{d}{dt}\log\rho = v^2\frac{dv}{dr} + v\frac{dv}{dt} - k\left(\frac{dv}{dr} + \frac{v}{r}\right) \text{.........} (2).$$

Eliminating ρ from (1) and (2) by differentiation, we obtain

$$\frac{d^2v}{dt^2} = k\left(\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right) - \frac{d}{dr}\left(\frac{dv^2}{dt} + v^2\frac{dv}{dr}\right).$$

(7) *A mass of liquid surrounds a solid sphere of radius* a, *and its outer surface, which is a concentric sphere of radius* b, *is subject to a given constant pressure* Π, *no other forces being in action on the liquid. The solid sphere suddenly shrinks into a concentric sphere; it is required to determine the subsequent motion, and the impulsive action on the sphere.*

At the time t, let p be the pressure and v' the velocity at the distance r'; then the equation of motion is

$$\frac{1}{\rho}\frac{dp}{dr'} = -\frac{dv'}{dt} - v'\frac{dv'}{dr'};$$

or, taking account of the equation of continuity,

$$r'^2 v' = F(t),$$

$$\frac{1}{\rho}\frac{dp}{dr'} = -\frac{F'(t)}{r'^2} - v'\frac{dv'}{dr'}.$$

Let R, r be the radii of the external and internal surfaces, and V, v their velocities; these quantities are functions of t only, and

$$V = \frac{dR}{dt}, \quad v = \frac{dr}{dt}.$$

Integrating the equation from r to R,

$$\frac{\Pi}{\rho} = \tfrac{1}{2}(v^2 - V^2) - F'(t)\left(\frac{1}{r} - \frac{1}{R}\right),$$

or $$\frac{\Pi}{\rho} = \frac{1}{2}v^2\left(1 - \frac{r^4}{R^4}\right) - \left(\frac{1}{r} - \frac{1}{R}\right)\left(r^2 v\frac{dv}{dr} + 2rv^2\right).$$

Putting $v^2 = z$, and multiplying by $2r^2$, and observing that $R^3 - r^3 = b^3 - a^3 = c^3$, this becomes

$$2\frac{\Pi r^2}{\rho} = zr^4\left\{\frac{1}{r^3} - \frac{r^2}{(c^3 + r^3)^{\frac{4}{3}}}\right\} - \left\{\frac{1}{r} - \frac{1}{(c^3 + r^3)^{\frac{1}{3}}}\right\}\frac{d}{dr}(zr^4).$$

Integrating, we obtain

$$\frac{2}{3}\frac{\Pi}{\rho}\frac{a^3-r^3}{r^4}=v^2\left(\frac{1}{r}-\frac{1}{R}\right).$$

Take r for the radius of the solid sphere inside, and let ϖ be the impulsive pressure at the distance r';

then $$d\varpi=-\rho v'\,dr'=-\rho\frac{r^2v\,dr'}{r'^2};$$

therefore, since $\varpi=0$, when $r'=R$,

$$\frac{\varpi}{\rho}=r^2v\left(\frac{1}{r}-\frac{1}{R}\right).$$

The whole impulse on the sphere

$$=4\pi r^2\varpi=4\pi\rho r^2v\frac{R-r}{R},$$

and the whole momentum destroyed

$$=\int_r^R 4\pi r'^2\rho v'\,dr'$$
$$=4\pi\rho r^2v\,(R-r);$$

these two quantities are therefore in the ratio of r to R.

The velocity may also be obtained at once by help of the principle of energy.

For the kinetic energy

$$=\tfrac{1}{2}\int_r^R 4\pi r'^2\rho v'^2\,dr'$$
$$=2\pi\rho\int\frac{r^4v^2\,dr'}{r'^2}=2\pi\rho r^4v^2\left(\frac{1}{r}-\frac{1}{R}\right),$$

and the work done by the outer pressure

$$=\int_b^R 4\pi R^2\Pi(-dR)$$
$$=\frac{4}{3}\pi\Pi(b^3-R^3)$$
$$=\frac{4}{3}\pi\Pi(a^3-r^3).$$

Hence the equation of energy gives us at once the expression for the velocity.

EXAMPLES.

1. The main of the water supply of a town is one foot in diameter, and side pipes an inch in diameter leave it at intervals successively; find the velocity in the main after any number of these side pipes have been passed; that before passing any of them being given, and the water being supposed to flow freely and steadily in all.

2. A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; shew that the equation of continuity is

$$\frac{d\rho}{dt}+\frac{d(\rho\omega)}{d\theta}=0,$$

where ω is the angular velocity of a particle whose vectorial angle, measured from a line in the plane of its motion through the fixed axis, is θ at the time t.

3. A mass of fluid is in motion so that the lines of motion lie on the surfaces of co-axial cylinders; find the equation of continuity.

4. Each particle of a mass of liquid moves in a plane through the axis of z; find the equation of continuity.

5. If r, θ, be the polar co-ordinates of a point at which the density is ρ, and u, v, the velocities along, and perpendicular to the radius vector, shew that the equation of continuity for motion in one plane, is

$$\frac{d(\rho ru)}{dr}+\frac{d(\rho v)}{d\theta}+r\frac{d\rho}{dt}=0.$$

6. The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is

$$\frac{d\rho}{dt}+V\frac{d\rho}{dr}+\frac{\rho}{r^2}\frac{d}{dr}(r^2V)=0,$$

V being the velocity at a distance r.

7. A cubical vessel, just filled with water, slides down a smooth inclined plane; find the whole pressure on any side. Also determine the whole pressure when the vessel slides down a rough inclined plane.

8. Two vessels, containing liquid, acted on by no forces, but subject to a given external pressure, are connected by a cylindrical tube of small bore. A portion of the liquid in the tube being supposed to be suddenly annihilated, determine the instantaneous change of pressure, and the subsequent motion in the tube.

9. A closed vessel, filled with air, is moved, in a vertical direction, with a given acceleration; find what the law of the density must be in order that the air may be at rest, relative to the vessel.

10. A closed vessel is filled with water containing in it a piece of cork which is free to move; if the vessel be suddenly moved forwards by a blow, shew that the cork will shoot forwards relatively to the water.

11. If M be the quantity of fluid at the time t inside a given closed surface S completely enclosed in the fluid, and if at any point of the surface ρ be the density, q the velocity of the fluid, and θ the angle the direction of the velocity makes with the outward-drawn normal to the surface S, prove that

$$\frac{dM}{dt} + \iint \rho q \cos\theta dS = 0,$$

where dS is an element of the surface.

12. A vertical cylindrical vessel, open at the top and containing water, is let fall from a given height on a horizontal plane; the vessel and water being supposed inelastic, find the impulsive pressure at any point at the instant of impact.

If a piece of cork be immersed, and kept under the surface by a string fastened to the base of the vessel, find the impulsive tension of the string.

13. A spherical shell the internal and external surfaces of which are concentric, and which is just filled with water, is placed on a smooth inclined plane and allowed to slide down; find the resultant vertical pressure on the internal surface.

If the shell be suddenly stopped at any time, find the impulsive pressure at any point of the fluid; and shew that the resultant impulses on the two portions into which the shell would be divided by the plane through its centre perpendicular to the inclined plane and perpendicular to the vertical plane of motion are in the ratio 5 : 1.

14. The bob of a pendulum is a hollow sphere filled with liquid; find the surfaces of equal pressure for any position of the pendulum.

15. A closed cubical box, very nearly filled with liquid, is placed on a smooth horizontal table, so that two of its faces are parallel to the edge of the table, and a string, passing over the edge and supporting a weight, is fastened to the middle point of the side of the base nearest the edge; determine the surfaces of equal pressure during the motion, and compare the pressures on the top and on the base of the box.

If a small sphere of greater density than the liquid be suspended in it by a string fastened to the top of the box, and another small sphere of less density than the liquid be attached by a string to the base of the box, find the directions of the strings when the spheres are in equilibrium relative to the liquid.

What would be the effect on these spheres of suddenly destroying the motion of the box?

16. A spherical shell, the internal surface of which is smooth, is just filled with liquid, and, being placed initially very near the highest point of a fixed rough sphere, is allowed to roll down; determine at any time the surfaces of equal pressure.

If the motion be at any instant suddenly arrested, compare the whole impulsive actions on the portions into which the internal surface is divided by a horizontal plane through its centre.

17. If a bombshell explode at a great depth beneath the surface of the sea, prove that the impulsive pressure at any point varies inversely as the distance from the centre of the shell.

18. A straight tube of small bore, ABC, is bent so as to make the angle ABC a right angle, and AB equal to BC. The end C is closed; and the tube is placed with the end A upwards and AB vertical, and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one half; and find the instantaneous change of pressure at any point of the horizontal tube.

19. A given mass of liquid in a cylindrical column is acted upon by a force parallel to the axis of the cylinder and varying as the distance from a fixed normal section (O) of the cylinder, and is kept at rest by a fixed plane (A) between the fluid and the end O. Find the pressure at any point, and if the plane A be suddenly removed, prove that the pressure at the central plane of the liquid mass is diminished in the ratio

$$h : h + 2c,$$

where $2h$ is the length of the column and c the distance of its central plane from O. Also prove that the pressure at any point of the liquid remains constant during the motion.

If there be a rigid plane at O, prove that when the motion is stopped the *whole* impulsive action on the surface of the cylinder is to the impulse on the plane O as the length of the column is to the radius of the cylinder.

20. Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the steam, and if the motion be supposed to be that of divergence from the vertex of the cone, prove that

$$\frac{v}{V} = \frac{D^2}{d^2} \cdot \epsilon^{\frac{v^2 - V^2}{2k}},$$

where k is the pressure divided by the density, and supposed constant.

21. A mass of liquid, moving in a straight tube of uniform bore under the action of no forces meets a piston, which, by compressing a spring, gradually reduces the liquid to rest: if p be the pressure (on a unit of area) at any point of the fluid, whose distance, from the extremity (E) not in contact with the piston, is x; shew that, at any time t,

$$p = \frac{P}{V} x,$$

V being the volume of the liquid, and P the pressure exercised by the piston, at the time t, upon the extremity (E') of the liquid in contact with it.

22. In the preceding problem, if the bore of the tube be variable and small, shew that

$$p = \frac{P\int_0^x X dx}{\kappa\int_0^u X dx},$$

where κ is the area of the piston, $\frac{1}{X}$ the area of the section of the tube at the distance x from (E), and u the distance of E' from E.

23. Prove that if the co-ordinates of a particle of a mass of liquid be given by the equations

$$x = h + ce^{-\frac{k}{c}} \sin\left(\omega t + \frac{h}{c}\right), \quad y = k + ce^{-\frac{k}{c}} \cos\left(\omega t + \frac{h}{c}\right),$$

where c and ω are constants and h and k constants differing for each particle, the equation of continuity is satisfied.

Prove also that the motion is rotational, and that the angular velocity is

$$= \frac{\omega}{\epsilon^{\frac{2k}{c}} - 1}.$$

24. A current of water flows through a fine tube so that the velocity is v; x, y, z and v are functions of s the length of the tube up to the point (x, y, z) and of the time: prove that the acceleration of a fluid particle in the direction of x is

$$\frac{d^2x}{dt^2} + 2v\frac{d^2x}{dsdt} + v^2\frac{d^2x}{ds^2} + \frac{dv}{dt}\frac{dx}{ds} + \frac{dv}{ds}\frac{dx}{dt}.$$

25. An elastic fluid, the weight of which is neglected, is in motion in a uniform straight tube; shew that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation,

$$\frac{d^2v}{dt^2} + \frac{d}{dr}\left(2v\frac{dv}{dt} + v^2\frac{dv}{dr}\right) = k\frac{d^2v}{dr^2}.$$

26. Air is in motion in a uniform tube of small section; prove that if ρ be the density and v the velocity at a distance x from a fixed point at the time t,

$$\frac{d^2\rho}{dt^2} = \frac{d^2}{dx^2}\{(v^2 + k)\rho\}.$$

27. A cylindrical vessel with its axis vertical contains liquid, and it is heated at the base in such a manner that the density at a point in the liquid at a distance x above the base at a time t, measured from a fixed epoch when the height of the liquid was c, is $\rho_0(1 + mt - nx)^{-1}$. If in the course of the motion which must necessarily ensue, no vapour is given off and the liquid is always continuous, prove that the velocity and pressure are given by the equations

$$v = \frac{mx}{1+mt}, \quad \frac{p}{\rho_0} = \frac{g}{n} \log \frac{1+mt-nx}{(1+mt)(1-nc)},$$

assuming that there is no pressure at the upper surface.

28. If the motion of elastic fluid be symmetrical with regard to a fixed point, and if v be the velocity at the time t of a particle at a distance r, prove that

$$\frac{d^2v}{dt^2} = \kappa\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} - \frac{2v}{r^2}\right) - \frac{d}{dr}\left(\frac{dv^2}{dt} + v^2\frac{dv}{dr}\right).$$

29. Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled, one with water, and the other with air of such a density as to support a column h of water, h being less than c. If a communication be opened between them at their bases, the height x, to which the water rises, is given by the equation

$$cx - x^2 + ch \log \frac{c-x}{c} = 0.$$

30. An infinite mass of liquid acted upon by no forces is at rest, and a spherical portion of radius c is suddenly annihilated; the pressure ϖ at an infinite distance being supposed to remain constant, prove that the pressure at the distance r from the centre of the sphere is instantaneously diminished in the ratio $r-c : r$, and that the cavity will be filled up in the time

$$\sqrt{\frac{\pi\rho c^2}{6\varpi}} \cdot \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)}.$$

31. A mass of liquid, not acted upon by any forces, is contained between two fixed parallel planes, and surrounds a solid cylinder of radius a connecting the two planes, the radius of the free surface being b, and the external pressure ϖ. If the solid cylinder be suddenly annihilated, prove that the instantaneous pressure at a distance r from the axis of the cylinder is to ϖ in the ratio

$$\log\frac{r}{a} : \log\frac{b}{a}.$$

Also find a linear differential equation for determining the velocity of collapse during the subsequent motion.

If b be infinitely large, prove that no motion will take place.

CHAPTER XI.

STEADY MOTION AND PARALLEL SECTIONS.

150. WHEN the motion of a fluid is such that the velocity of a fluid particle is a function of the co-ordinates x, y, z only, and does not involve the time explicitly, that is, when the velocities of the particles of fluid which pass in succession through a given point are always the same, the motion is characterised as *steady motion*.

On this hypothesis, or, in other words, in the cases for which such a motion is possible, the expressions $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{dw}{dt}$, do not appear in the original equation, and $\frac{d\phi}{dt}$ will therefore not appear in the final equation, which is, in consequence, (Art. 141),

$$\frac{1}{\rho}\frac{dp}{ds} = S - v\frac{dv}{ds},$$

employing v to represent the velocity.

As an instance of steady motion, consider the case of a vessel kept constantly full of water, and having a horizontal orifice in its base, from which the water issues at an uniform rate. The vessel may be supposed to be in the form of a surface of revolution, and to have its base horizontal.

Gravity being the only force in action, $S = g\frac{dz}{ds}$, if z be measured vertically downwards from the free surface, and

$$\frac{1}{\rho}\frac{dp}{ds} = g\frac{dz}{ds} - v\frac{dv}{ds};$$

$$\therefore \frac{p}{\rho} - gz + \tfrac{1}{2}v^2 = C.$$

Let U be the velocity at the surface, and u at the orifice; then, taking h as the depth of the orifice below the surface, and Π as the atmospheric pressure,

$$\frac{\Pi}{\rho} = C - \tfrac{1}{2}U^2,$$

$$\frac{\Pi}{\rho} = gh + C - \tfrac{1}{2}u^2,$$

$$\therefore\ u^2 = U^2 + 2gh.$$

But, if A be the area of the surface, and K of the orifice, and if the motions of all the issuing particles be supposed perpendicular to the plane of the orifice,

$$AU = Ku,$$

since the quantity of water poured in at the surface in any time is equal to the quantity which passes through the orifice in the same time;

$$\therefore\ U = \sqrt{(2gh)}\,\frac{K}{\sqrt{(A^2 - K^2)}},$$

$$\text{and } u = \sqrt{(2gh)}\,\frac{A}{\sqrt{(A^2 - K^2)}}.$$

If the orifice be very small, the ratio $\frac{K}{A}$ may be neglected, and, approximately, $u = \sqrt{(2gh)}$.

Suppose the orifice not in the base of the vessel, and so small that the velocities of all the particles passing through it are sensibly the same; we then have, as in the previous case,

$$u^2 = U^2 + 2gh,\quad AU = Ku,$$

and approximately, $u = \sqrt{(2gh)}$.

If the vessel be not kept constantly full, the motion will not be steady, but when the orifice is very small, it may be taken as being approximately steady, and the expression $\sqrt{(2gh)}$ may be employed as the velocity of the issuing liquid.

Looking upon the issuing liquid as a series of particles in motion under the action of gravity, every particle moves in a

parabolic path, and the issuing liquid takes the form of a parabolic arc. Moreover since the velocity at the orifice is, approximately, that due to the height h, the directrix of the parabola is approximately coincident with the surface of the liquid.

151. *To find the time in which a given quantity of liquid will flow through a small orifice.*

At the time t, let x be the height of the surface above the orifice, and X its area.

Then, approximately,

$$\text{velocity at the orifice} = \sqrt{(2gx)}:$$

$$\text{but } -\frac{dx}{dt} \text{ is the velocity of the surface,}$$

$$\therefore\ -X\frac{dx}{dt} = \kappa\sqrt{(2gx)},$$

$$\text{or } \frac{dt}{dx} = -\frac{X}{\kappa\sqrt{(2gx)}}.$$

X being a known function of x, this equation gives t in terms of x, and therefore x in terms of t.

It will be seen hereafter that, in certain cases, particularly when the containing vessel is formed of a thin substance, a considerable modification of the value of κ, employed in the preceding process, is requisite, in order to obtain results in approximate accordance with observations.

Ex. 1. *A hollow cone, having its axis vertical, is filled with water; required to find the time in which it will be emptied through a small aperture at its vertex.*

In this case, $X = \pi x^2 \tan^2\alpha$, taking 2α as the vertical angle;

$$\therefore\ \frac{dt}{dx} = -\frac{\pi\tan^2\alpha}{\kappa\sqrt{(2g)}}x^{\frac{3}{2}},$$

and, if h be the height of the cone, the time (t) in which it will be emptied is

$$-\int_h^0 \frac{\pi\tan^2\alpha}{\kappa\sqrt{(2g)}}x^{\frac{3}{2}}dx, \text{ or } \frac{1}{5}\frac{\pi h^2\tan^2\alpha}{\kappa}\sqrt{\left(\frac{2h}{g}\right)}.$$

If the cone had been kept constantly full, the velocity at the orifice would have been always $\sqrt{(2gh)}$, and the same quantity of liquid would have flowed out in a time τ, such that

$$\tau\kappa\sqrt{(2gh)} = \tfrac{1}{3}\pi h^3 \tan^2\alpha;$$

hence we obtain $t : \tau :: 6 : 5$.

Ex. 2. *A vessel, in the form of a surface of revolution, has a small aperture at its lowest point; determine its form so that the surface of water, contained in it, may descend uniformly.*

We must have $\frac{dx}{dt}$ constant, and therefore $\frac{X}{\sqrt{x}}$ constant; but, if $y = f(x)$ be the generating curve, $X = \pi y^2$, and therefore $\frac{y^2}{\sqrt{x}}$ is constant: hence the generating curve is one of the class

$$y^4 = a^3 x,$$

the velocity of descent being determined by the value of a.

This example contains the theory of the Clepsydra or water-clock.

Ex. 3. *To find the rate of efflux through a small orifice in the base of a vessel in motion in a vertical direction.*

If the orifice be very small compared with the upper surface of the fluid, we may suppose during any small time that the motion is relatively steady, and therefore that the relative motion of the fluid would be the same as in a vessel at rest, if the quantity g be replaced by $g + f$, f being the vertical acceleration.

We thus obtain, if h be the depth of the orifice, and v the velocity of efflux, relative to the vessel,

$$v^2 = 2(g + f)h.$$

The Hypothesis of Parallel Sections.

152. Suppose the interior of a vessel to be a surface of revolution, the axis of which is vertical; and suppose moreover that the inclination to the vertical of the generating curve is always small, and does not change rapidly.

If such a vessel contain liquid, which is allowed to flow out through a horizontal aperture in its base, it is evident that the particles will move in directions nearly vertical, and the velocities of all particles in the same horizontal plane will be very nearly the same. The discussion of the real motions in such a case would be excessively complicated, but an approximate solution may be obtained by means of the hypothesis, that the successive horizontal laminæ descend vertically, and replace each other in succession, that is, that the motions of all the particles in a horizontal plane are the same, and all vertical.

This is the hypothesis of parallel sections, and it is clearly equivalent to the neglecting of all horizontal motions, and of the changes of relative position which take place in the component particles of the descending laminæ of liquid.

If the orifice be much less than the horizontal base of the vessel, the motions of the particles near the base cannot be all vertical, and the same, in the same horizontal plane; the hypothesis therefore will not even approximately hold good. In order however to obtain a solution of the question, the hypothesis will be made throughout, and a large allowance must therefore be made for the probable error arising from this cause.

Under this head we shall discuss the following cases.

153. *A vase in the form of a surface of revolution, and having a finite horizontal aperture in its base, is kept constantly full; required to determine the rate at which liquid must be poured in.*

Let A be the area of the top of the vase, K of the aperture, and h the depth of the vase.

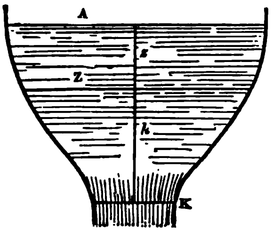
At a depth z below the surface, where Z is the area of the horizontal section, let v be the velocity at the time t, and, at the same time, let U be the velocity at the surface and u at the aperture.

Since the same quantity must pass through any horizontal section in the same element of time δt,

$$U\delta t \,.\, A = u\delta t \,.\, K = v\delta t \,.\, Z,$$

$$\text{or } AU = Ku = Zv.$$

These conditions, it will be observed, express the *continuity* of the fluid.



The equation of motion is

$$\frac{1}{\rho}\frac{dp}{dz} = g - \frac{dv}{dt} - v\frac{dv}{dz}.$$

U and u are functions of t; Z is a function of z; and, from the equation

$$v = \frac{AU}{Z}, \quad \frac{dv}{dt} = \frac{A}{Z}\frac{dU}{dt};$$

$$\therefore \; \frac{1}{\rho}\frac{dp}{dz} = g - \frac{A}{Z}\frac{dU}{dt} - v\frac{dv}{dz},$$

$$\text{and } \frac{p}{\rho} = C + gz - A\frac{dU}{dt}\int\frac{dz}{Z} - \tfrac{1}{2}v^2,$$

where C may be a function of t.

Let Π be the pressure at the surface;
then, when $z = 0$, $p = \Pi$, $v = U$,

$$\text{and } \frac{p - \Pi}{\rho} = gz - A\frac{dU}{dt}\int_0^z \frac{dz}{Z} - \tfrac{1}{2}(v^2 - U^2)$$

$$= gz - A\frac{dU}{dt}\int_0^z \frac{dz}{Z} - \tfrac{1}{2}U^2\left(\frac{A^2}{Z^2} - 1\right).$$

Let Π' be the pressure at the orifice;
putting $z = h$, and therefore $Z = K$,

$$\frac{\Pi' - \Pi}{\rho} = gh - A\frac{dU}{dt}\int_0^h \frac{dz}{Z} - \tfrac{1}{2}U^2\left(\frac{A^2}{K^2} - 1\right).$$

If the vase be in air Π' and Π will be sensibly the same, and, assuming this to be the case, we have, for the determination of U, the equation

$$A\frac{dU}{dt}\int_0^h \frac{dz}{Z} = gh - \frac{1}{2}\left(\frac{A^2}{K^2} - 1\right)U^2.$$

Let $A\int_0^h \frac{dz}{Z} = a$, and $\frac{A^2}{K^2} - 1 = 2m$;

$$\text{then } a\frac{dU}{dt} = gh - mU^2,$$

$$dt = \frac{a\,dU}{m\left(\frac{gh}{m} - U^2\right)};$$

$$\therefore \frac{m}{a}t = \frac{1}{2}\sqrt{\left(\frac{m}{gh}\right)}\log\frac{\sqrt{\left(\frac{gh}{m}\right)} + U}{\sqrt{\left(\frac{gh}{m}\right)} - U} + C',$$

$$\text{or, } \frac{\sqrt{\left(\frac{gh}{m}\right)} + U}{\sqrt{\left(\frac{gh}{m}\right)} - U} = C\epsilon^{\frac{2\sqrt{(ghm)}}{a}t};$$

$$\therefore U = \sqrt{\left(\frac{gh}{m}\right)} \cdot \frac{C\epsilon^{\alpha t} - 1}{C\epsilon^{\alpha t} + 1}, \text{ if } 2\sqrt{(ghm)} = a\alpha$$

$$= \sqrt{\left(\frac{gh}{m}\right)} \cdot \frac{C - \epsilon^{-\alpha t}}{C + \epsilon^{-\alpha t}}.$$

Suppose that initially the vase was just filled, and the liquid then allowed to escape at the orifice, the vase being kept full by pouring in liquid above; then initially $U = 0$;

$$\therefore U = \sqrt{\left(\frac{gh}{m}\right)}\frac{1 - \epsilon^{-\frac{2\sqrt{(ghm)}}{a}t}}{1 + \epsilon^{-\frac{2\sqrt{(ghm)}}{a}t}};$$

this equation determines the rate at which liquid is being poured in at the time t.

The quantity which has been poured in from the beginning of the motion to the time t'

$$= \int_0^{t'} UA\,dt$$

$$= A\sqrt{\left(\frac{gh}{m}\right)}\int_0^{t'} \frac{1 - \epsilon^{-\alpha t}}{1 + \epsilon^{-\alpha t}}\,dt$$

$$= A\sqrt{\left(\frac{gh}{m}\right)}\int_0^{t'} \left\{1 - \frac{2\epsilon^{-\alpha t}}{1 + \epsilon^{-\alpha t}}\right\}dt$$

$$= A\sqrt{\left(\frac{gh}{m}\right)}\left\{t' + \frac{2}{\alpha}\log\frac{1 + \epsilon^{-\alpha t'}}{2}\right\}.$$

If the motion be continued for a long period of time, we observe that U approximates to a 'terminal velocity' $\sqrt{\left(\frac{gh}{m}\right)}$,

$$\text{or, } \sqrt{\left(2gh \cdot \frac{K^2}{A^2 - K^2}\right)}.$$

$$\text{Also, } u = \frac{AU}{K} = \sqrt{\left(2gh \frac{A^2}{A^2 - K^2}\right)} \frac{1 - \epsilon^{-at}}{1 + \epsilon^{-at}},$$

and approximates to a terminal value,

$$\sqrt{\left(2gh \frac{A^2}{A^2 - K^2}\right)}.$$

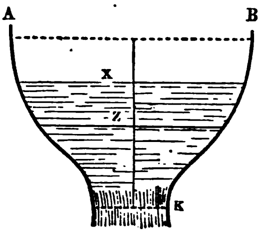
If K is small compared with A, these are, approximately,

$$\frac{K}{A}\sqrt{(2gh)}, \text{ and } \sqrt{(2gh)};$$

results which might have been anticipated, for it is clear that ultimately the motion will become 'steady.'

154. *A vase, having a horizontal aperture in its base, contains liquid, which is allowed to flow out through the orifice; required to determine the motion.*

At the time t, let x be the vertical space through which the surface of the liquid has descended from its original level AB,



X the area of the section at the surface,

Z the area of the section at the depth z below AB,

U the velocity at the surface, u at the orifice, and v at the level of z, the equation of motion is

$$\frac{1}{\rho}\frac{dp}{dz} = g - \frac{dv}{dt} - v\frac{dv}{dz}.$$

$$\text{Also} \qquad Ku = XU = Zv,$$

where u is a function of t, X of x, Z of z, v of z and t, x of t, and $U = \frac{dx}{dt}$;

$$\therefore \frac{dv}{dt} = \frac{K}{Z}\frac{du}{dt},$$

and $$\frac{1}{\rho}\frac{dp}{dz} = g - \frac{K}{Z}\frac{du}{dt} - v\frac{dv}{dz},$$

and $$\frac{p}{\rho} = C + gz - K\frac{du}{dt}\int\frac{dz}{Z} - \tfrac{1}{2}v^2$$

$$= C + gz - K\frac{du}{dt}\int\frac{dz}{Z} - \frac{1}{2}\frac{K^2u^2}{Z^2}.$$

At the time t, when $z = x$,

$$p = \Pi \text{ and } Z = X,$$

and, h being the depth below AB of the orifice K, when $z = h$, $p = \Pi$ and $Z = K$,

$$\therefore 0 = g(h - x) - K\frac{du}{dt}\int_x^h \frac{dz}{Z} - \tfrac{1}{2}K^2u^2\left(\frac{1}{K^2} - \frac{1}{X^2}\right).$$

Now $$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = U\frac{du}{dx} = \frac{Ku}{X}\frac{du}{dx},$$

$$\therefore \frac{K^2}{X}u\frac{du}{dx}\int_x^h \frac{dz}{Z} + \tfrac{1}{2}u^2\left(1 - \frac{K^2}{X^2}\right) = g(h - x),$$

an equation of the form

$$\frac{du^2}{dx} + \alpha u^2 = \beta(h - x),$$

which determines u and therefore U in terms of x, and, from the equation

$$\frac{dx}{dt} = U,$$

we can obtain t in terms of x, and therefore x in terms of t.

The quantity of fluid which has escaped in the time t from the beginning of the motion is the volume of the vase between AB and X, that is,

$$\int_0^x Zdz.$$

It may also be expressed as the quantity which has flowed through the orifice in the time t, which

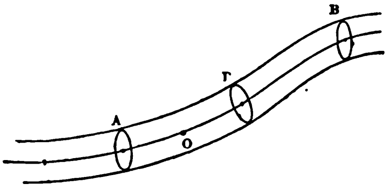
$$= \int_0^t Kudt.$$

Consequently $\quad \int_0^x Zdz = K\int_0^t udt,$
observing that x is a function of t.

As before, if K is very small compared with the values of Z, $\frac{K^2}{X^2}$ and $\frac{K^2}{X}\int_x^h \frac{dz}{Z}$ may be neglected, and, as a rough approximation, we have $u^2 = 2g(h-x)$.

155. III. *The motion of a liquid in a tube of small section.*

We shall suppose that the particles of liquid in any normal section move perpendicularly to the section, and that the volume of liquid is given.



Let O be a fixed point in the axis of the tube, AB the liquid in motion, Z the area of the section at a point P in the axis of the tube.

Take $AO = a$, $OB = a'$, $OP = s$, and let $mf'(s)$ be the force at P in direction of the axis on a particle m;

$$\therefore \frac{1}{\rho}\frac{dp}{ds} = f'(s) - \frac{dv}{dt} - v\frac{dv}{ds},$$

where v is the velocity at P.

If κ be the area of the section at O, and u the velocity,

$$\kappa u = Zv,$$

where v is a function of s and t, u of t, and Z of s;

$$\frac{dv}{dt} = \frac{\kappa}{Z}\frac{du}{dt} \text{ and } \frac{dv}{ds} = -\frac{\kappa u}{Z^2}\frac{dZ}{ds},$$

$$\therefore \frac{1}{\rho}\frac{dp}{ds} = f'(s) - \frac{\kappa}{Z}\frac{du}{dt} + \frac{\kappa^2 u^2}{Z^3}\frac{dZ}{ds},$$

$$\frac{p}{\rho} = C + f(s) - \kappa\frac{du}{dt}\int\frac{ds}{Z} - \frac{1}{2}\frac{\kappa^2 u^2}{Z^2}.$$

Let A, A' be the areas of the sections A and B, and take the pressures at A and B to be equal, then

$$0=f(\alpha')-f(-\alpha)-\kappa\frac{du}{dt}\int_{-\alpha}^{\alpha'}\frac{ds}{Z}-\frac{1}{2}\kappa^2u^2\left(\frac{1}{A'^2}-\frac{1}{A^2}\right)\text{.........}(1).$$

If V be the given volume,

$$V=\int_{-\alpha}^{\alpha'}Zds,$$

which gives α' in terms of α and therefore A' as well as A in terms of α.

Moreover $$\kappa u=-A\frac{d\alpha}{dt}\text{.......................}(2).$$

The equations (1) and (2) determine u and α in terms of the time.

156. *The motion of heavy liquid in an uniform tube of small section.*

In this case v is the same at all points of the tube, and therefore $\frac{dv}{ds}=0$, and the equation of motion is

$$\frac{1}{\rho}\frac{dp}{ds}=-g\frac{dz}{ds}-\frac{dv}{dt};$$

$$\therefore\ \frac{p}{\rho}=C-gz-\frac{dv}{dt}s.$$

Taking α and α' as the extreme values of s, γ and γ' of z, we obtain

$$0=g(\gamma'-\gamma)+\frac{dv}{dt}(\alpha'-\alpha),$$

or, if l be the length of the filament of fluid,

$$l\frac{dv}{dt}=-g(\gamma'-\gamma),$$

observing that $$\frac{dv}{dt}=\frac{d^2\alpha}{dt^2}=\frac{d^2\alpha'}{dt^2}.$$

Ex. *Liquid rests in a fine tube, the axis of which is a circle, of radius a, in a vertical plane.*

Let the filament subtend an angle 2α at the centre, and at the time t let θ be the angular distance from the vertical of the middle point of the filament.

Then $\gamma' - \gamma = a\{\cos(\alpha - \theta) - \cos(\alpha + \theta)\} = 2a \sin\alpha \sin\theta$,

$$l = 2a\alpha, \text{ and } \frac{dv}{dt} = a\frac{d^2\theta}{dt^2},$$

$$\therefore\ a\alpha\frac{d^2\theta}{dt^2} = -g\sin\alpha\sin\theta,$$

and, if the original displacement be small, the time of a small oscillation is

$$2\pi\sqrt{\left(\frac{a\alpha}{g\sin\alpha}\right)}.$$

157. *A vessel having a horizontal aperture in its base is partially immersed in a liquid of unlimited extent, and is kept constantly full of the same liquid.*

Let a be the height of the surface in the vessel above the surface of the external liquid, and h the depth of the aperture below the upper surface.

Measuring z from the upper surface, the equation of steady motion is

$$p = g\rho z - \tfrac{1}{2}\rho v^2 + C;$$

but, when $z = 0$, $p = \Pi$, and, when $z = h$, $p = \Pi + g\rho(h - a)$, therefore, if u be the velocity at the upper surface and u' at the aperture,

$$u'^2 = u^2 + 2ga.$$

If K, K' be the areas of the surface and the aperture, $Ku = K'u'$, and the quantity poured in during the unit of time

$$= Ku = KK'\sqrt{\left(\frac{2ga}{K^2 - K'^2}\right)}.$$

158. If there be a *finite vertical orifice* in the side of a vessel containing liquid, the rate of efflux can be calculated, when the motion is steady, by supposing the orifice to consist of a number of very small orifices, and by determining the aggregate of the effluxes through all the orifices.

Thus, if u be the velocity at the surface, and v at an element of the orifice κ, the depth of which is z,

$$v^2 = u^2 + 2gz,$$

and taking K as the area of the surface,

$$Ku = \Sigma\,(\kappa v),$$

or, if y be the breadth of the orifice at the depth z,

$$Ku = \int_a^b y\sqrt{(u^2 + 2gz)}\,dz,$$

a and b being the depths of the upper and lower boundaries of the orifice.

If the motion be not steady, an approximate solution can be obtained when the orifice, although finite, is not large, by supposing the motion steady during any elementary interval of time, and taking, as in the previous case, the sum of the quantities of liquid passing through all the small orifices into which the whole aperture is divided.

159. *A vessel in the form of a frustum of a surface of revolution with its axis vertical, and vertex downwards, contains water, which is flowing through the lower end. If the lower end be suddenly closed it is required to find the impulsive pressure at any point.*

Let U be the velocity, and A the area of the upper surface, v the velocity and Z the area of a horizontal section at the depth z; then, if ϖ be the impulsive pressure at the depth z,

$$\delta\varpi = \rho v \delta z.$$

Also $$U\,.\,A = v\,.\,Z,$$

$$\therefore\ \frac{d\varpi}{dz} = \frac{\rho U A}{Z},$$

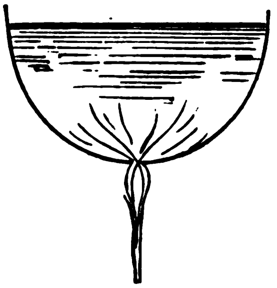
and the integration of this equation determines ϖ.

The contracted vein.

160.. When liquid issues through a small orifice in the thin base of a vessel, it is observed that the issuing stream is not cylindrical, but, near the orifice, is contracted so that its sectional area is less than the area of the orifice. The stream then expands and afterwards, as it descends, again diminishes gradually in size.

The sudden diminution of the issuing stream forms what is called the 'contracted vein,' and is due to the oblique or nearly horizontal motions of the particles near the edges of the orifice just before the efflux.

The after contraction, which is gradual, is due to the law of continuity, which requires that the mean velocity of the par-



ticles in any horizontal section of the issuing stream should vary inversely as the area of the section, and therefore that, as the velocity increases in the descent, the area of the section should diminish.

The discrepancy which exists between the results of theory and experiment is to a great extent accounted for by the contraction of the vein or filament of issuing liquid, and it is found moreover, as would be anticipated, that the amount of difference depends upon the nature of the orifice.

For instance, if the orifice be simply an opening in the side of the vessel, and if the side be very thin, the quantity of liquid which flows out in a given time is about $\frac{5}{8}$ths of the quantity given by the theory. Again, when the liquid issues through a cylindrical aperture of sensible length, formed by attaching to the orifice, externally, a small hollow cylinder, the ratio is found to be about $\frac{4}{5}$ths; but, if the cylinder be attached internally, the rate of efflux is about one half the theoretical rate*.

The *rate of efflux* depends upon the area of the orifice and the velocity of the issuing stream; it it shewn by experiment that the latter is, in general, not very different from the theo-

* Poisson, *Mécanique*, Art. 676.

retical velocity, and the observed error in the rate of efflux is therefore to a great extent accounted for by the formation of the 'contracted vein.'

An account of experiments, made by Bossut and others, on the efflux of liquids through orifices of various kinds, is given in the Encyclopædia Metropolitana, *Hydrodynamics*, p. 207.

Motion of Elastic Fluids.

161. If elastic fluid move in a tube the section of which does not change rapidly in size, we may make use of the hypothesis of parallel sections as before.

Assuming the motions of all the particles in any one section to be sensibly in the same direction, parallel to the axis of the tube, and neglecting gravity, the action of which will not sensibly affect the pressure, the equation of motion is

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{dv}{dt} - v\frac{dv}{dx},$$

where v is the velocity at the time t in a section at a distance x from a fixed section.

The equation of continuity, depending on the hypothesis which neglects all motions but those perpendicular to the section, is determined as follows.

Let X be the area of the section at a distance x, and ρ the density about this section at the time t, so that ρ is a function of x and t.

Then $\rho vX\delta t$ is the mass of fluid which flows across the section in the time δt;

$$\therefore \ \{\rho vX + \frac{d}{dx}(\rho vX)\,\delta x\}\,\delta t$$

is the quantity which flows across the section defined by the distance $x + \delta x$, and $-\frac{d}{dx}(\rho vX)\,\delta x\,\delta t$ is the increase of the quantity of fluid in the volume $X\delta x$ during the time δt, which is also given by the expression

$$X\delta x\,.\left(\frac{d\rho}{dt}\,\delta t\right);$$

$$\therefore\ X\frac{d\rho}{dt}+\frac{d}{dx}(\rho vX)=0$$

is the equation of continuity.

We have also, if the temperature remain constant,

$$p=k\rho,$$

and our equations become

$$\left.\begin{aligned}\frac{k}{p}\frac{dp}{dx}+\frac{dv}{dt}+v\frac{dv}{dx}&=0,\\ X\frac{dp}{dt}+\frac{d}{dx}(pvX)&=0.\end{aligned}\right\}$$

We shall not discuss the system of partial differential equations thus obtained, but proceed to consider the particular case in which the motion is steady.

It may be supposed that the air is supplied from a large reservoir at a constant pressure, and we shall then have

$$\frac{dv}{dt}=0,\quad \frac{dp}{dt}=0,$$

and
$$\left.\begin{aligned}\therefore\ \frac{k}{p}\frac{dp}{dx}+v\frac{dv}{dx}&=0,\\ \frac{d}{dx}(pvX)&=0,\end{aligned}\right\}$$

or
$$\left.\begin{aligned}k\log p+\tfrac{1}{2}v^2&=C\\ pvX&=C'\end{aligned}\right\}.$$

Let x be measured from a plane in which the pressure is sensibly the constant pressure, Π', of the reservoir, and let A be the area of the section, and U the velocity of the particles in it.

Also let u be the velocity of efflux,

K the area of the orifice, and Π the pressure.

$$\therefore\ \tfrac{1}{2}(u^2-U^2)+k\log\frac{\Pi}{\Pi'}=0,$$

or
$$u^2=U^2+2k\log\frac{\Pi'}{\Pi},$$

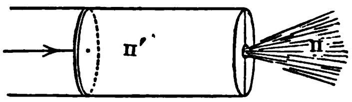
and
$$\Pi uK=\Pi' UA;$$

$$\therefore\ u^2\left(1-\frac{\Pi^2K^2}{\Pi'^2A^2}\right)=2k\log\frac{\Pi'}{\Pi}.$$

If U be very small, or K small compared with A, we have approximately

$$u^2=2k\log\frac{\Pi'}{\Pi}.$$

Suppose the air to be forced out of a cylinder through a small orifice by a piston moving slowly and exerting a constant pressure. The piston moving slowly with a velocity U, we may assume the motion as approximately steady;



$$\therefore\ k\log p+\tfrac{1}{2}v^2=C;$$

and, as before, $$u^2-U^2=2k\log\frac{\Pi'}{\Pi},$$

gives the velocity (u) of efflux.

162. From the equation of steady motion for elastic fluid, not under the action of any force,

$$\frac{k}{p}\frac{dp}{ds}=-v\frac{dv}{ds},$$

we obtain $k\log p=C-\frac{1}{2}v^2$, or $p=\Pi\epsilon^{-\frac{v^2}{2k}}$,

Π being determined by knowing the pressure for a given velocity.

It follows therefore that p is diminished by an increase of velocity, a theoretical result which can be easily verified by experiment.

One form of the experiment is as follows. To one end of a straight tube let a plane disc be fitted which is capable of sliding on wires projecting from the end of the tube; if the disc be placed at a small distance from the end, and a person blow steadily into the tube, the disc will be *drawn* towards the tube, and, instead of being blown off the wires, will oscillate slightly about a position very near the end of the tube.

Or the experiment may be more simply performed by fastening a straw with sealing-wax to a piece of card-board having a small hole in it. If a piece of paper be placed over the hole and the experimenter blow through the straw, the paper will bend so as to allow the egress of the air, but will not be detached from the card.

The history of this experiment, and the variations which occur in practice for different sizes of the aperture and the disc, are given by Professor Willis, in the *Cambridge Philosophical Transactions*, Vol. III. Part I. The fact was first observed in some iron works in France, about 1826, where one of the forge-bellows opened in a flat wall, and it was found that a board presented to the blast was sucked up against the wall. An experiment was however devised by Hawksbee, in 1719, which is equally illustrative of the theory. Hawksbee's experiment simply consisted in passing a current of air through a small box, and he observed that the air contained in the box became considerably rarified, a fact in accordance with the result that, neglecting changes of temperature, the pressure, and therefore also the density, is diminished by an increase of velocity.

In the preceding investigations on the motion of elastic fluids, the temperature has been considered uniform; if, however, the motion be very rapid, a sensible change of temperature takes place, and the results obtained must therefore, in such cases, be subject to considerable modification.

It may be here noticed, that an experiment, similar to the foregoing, was performed by M. Hachette, in 1826, with a stream of water and with a similar result. The explanation is the same; that is, it appears from the equation of steady motion, for incompressible fluids, that the pressure diminishes with an increase of velocity.

In the monthly numbers of *Nature* for November and December, 1875, Mr W. Froude has given an explanation, in general terms, and a series of experimental illustrations, of the fact that an increase of kinetic energy in a liquid is accompanied by a diminution of pressure.

EXAMPLES.

1. Find the time of emptying a paraboloid of revolution with its axis vertical and vertex downwards, through a small orifice at the vertex.

2. Shew that the time in which a cone, the axis of which is inclined to the vertical, will be emptied through a hole at the vertex, is $At \div 5\kappa$, where A is the area of the surface of the fluid at first, κ of the orifice, and t is the time of falling freely through the entire vertical space described by the fluid.

3. A vessel in the form of a surface of revolution, the axis of which is vertical, has a small orifice at its vertex, and is filled with fluid; determine its form in order that the quantity of fluid which flows out in any time may vary as the square root of the time.

4. A circular orifice is made in the horizontal base of a vessel containing fluid; if the fluid in the vessel is constantly kept at the same height, the descending stream is bounded by the surface generated by the revolution of the curve $y^4x = \text{const.}$, about the axis of x.

5. A right circular cone, with axis vertical and vertex upwards, is initially full of fluid which escapes by a small orifice in the curved surface; assuming the motion to be approximately steady, determine

(1) The time of emptying;

(2) The locus of the foci of the parabolic paths described by individual particles of fluid;

(3) The mean height above the orifice reached by the jet during the motion.

6. Liquid which occupies a length $2c$ of a fine tube of uniform bore, whose equation is $x = f(s)$, the origin being on the curve, the axis of x vertical, and the curve symmetrical with regard to the axis of x, rests in stable equilibrium with the middle point of the filament at the origin: shew that, if it be slightly disturbed, the time of a small oscillation is $\pi\sqrt{\dfrac{c}{gf'(c)}}$.

7. The side of a vessel containing fluid is a plane inclined to the vertical, and small orifices are made along its line of intersection with a vertical plane at right angles to it; prove that all the parabolic jets are touched by two fixed straight lines.

8. A vessel of the form of a slender parallelopiped is filled with fluid, and placed upon a rough horizontal plane; determine at what height a given orifice must be made in one of its vertical sides, in order that the issuing jet may have the greatest tendency to overthrow the vessel.

9. In the vertical side of a vessel containing fluid an infinite number of small holes, bored perpendicular to the side, lie in a straight line inclined at an angle $\tan^{-1}\frac{1}{2}$ to the horizon: find the equation to the surface of the issuing fluid, and shew that any horizontal section of it is a circle.

10. A vessel in the form of a frustum of a cone, with its axis vertical and wider end uppermost, contains water, which is flowing out through the lower end. If the lower end be suddenly closed, find the impulse at any point of the liquid, and the principal impulsive tensions at any point of the vessel.

11. A vessel in the shape of a surface of revolution, with its axis vertical, is filled with liquid. If a small hole be made at the vertex, it is found that the time of emptying the vessel is proportional to the time in which a heavy particle would fall from the surface of the liquid to the vertex; find the form of the vessel. Also find the form of the vessel when the time of emptying is proportional to the square root of the volume of liquid.

12. A filament of fluid oscillates in a thin cycloidal tube of uniform bore, the axis of the cycloid being vertical and vertex downwards. Supposing the filament to be placed initially with its lower end at the lowest point of the tube, find the pressure at any point of the filament at any time.

13. A vertical cylindrical vessel full of fluid has a fine crack extending along a generating line of the cylinder; find the time of emptying a given portion of the cylinder. Examine the case in which the time of emptying the whole cylinder is required.

14. A fine tube, the axis of which is in the form of a semicircle, stands in a vertical plane with its ends in the same hori-

zontal line, and filled with liquid. If one half of the liquid on one side of the lowest point be suddenly annihilated, find the initial change of pressure at any point of the other half.

15. A conical wine-glass has a fine crack along a generating line in the form of a triangle of breadth κ at the upper edge; shew that the glass would be emptied in a time $= \frac{15}{2} \frac{\pi \sin^2 \alpha}{\kappa \cos \alpha} \sqrt{\frac{a^5}{2g}}$:—where $2\alpha =$ vertical angle of the cone, and $a =$ length of the axis.

16. A vessel in the form of a surface of revolution is full of water, and it is noticed that if it be punctured horizontally at any point, the water flowing out strikes the horizontal plane on which the vessel was placed at the same distance from the axis of the vessel. Find the form of the vessel.

17. A right cone is filled with fluid and placed with a generating line horizontal, and uppermost, and a small orifice is made at the lowest point; find the time in which it will be emptied.

18. The surface of a vertical cylinder is pierced by a series of small holes in the form of a helix, the highest hole being at the top of the cylinder, and vertically above the lowest, and no other two holes being in the same vertical line. Determine the equation to the curve traced by the issuing fluid upon the horizontal plane passing through the lowest hole, the cylinder being kept constantly full.

Shew that the mean range is to the height of the cylinder as $\pi : 4$, and that the area included between the base of the cylinder and the curve above mentioned is

$$\frac{\pi h^2}{4}\left(\cot \alpha + \frac{8}{3}\right),$$

where α is the inclination of the line of holes to the horizon, and h the height of the cylinder.

19. An uniform semicircular tube stands in a vertical plane with its open ends resting in a vessel of fluid. One-third of its length is occupied with air, and the remainder with the fluid. Find the time of a small vibration caused by an instantaneous increase of the pressure of the fluid, considering the density of the air in the tube at any time to be uniform.

20. A filament of fluid oscillates in a thin hypocycloidal tube of uniform bore under the action of a force tending to the centre of the fixed circle, and varying as the distance: supposing the filament to be placed initially with one end at the vertex of the hypocycloid, find the pressure at any point of the filament at any time.

21. A small orifice of area κ is opened in the base of a vertical cylinder initially full of fluid. The fluid is forced through the orifice by a piston fitting the cylinder, to which is applied an uniform pressure P equal in amount to n times the weight of the fluid which the cylinder can contain. Shew that $\frac{1}{m}$th of the fluid will be evacuated in a time expressed by

$$2\left(\frac{A^2-\kappa^2}{\kappa^2}\cdot\frac{h}{2g}\right)^{\frac{1}{2}}\left\{\sqrt{(n+1)}-\sqrt{\left(n+1-\frac{1}{m}\right)}\right\},$$

where h is the height of the cylinder and A the area of its transverse section.

22. If the orifice of a conical vessel containing water be a section of the cone, perpendicular to its axis and at a distance δ from its vertex, and v be the velocity with which the water discharges itself, when its surface is at a distance z from the cone's vertex, prove that

$$\frac{dv^2}{dz}+\frac{(z+\delta)(z^2+\delta^2)}{z\delta^2}v^2-2g\frac{z^3}{\delta^3}=0,$$

the axis of the vessel being vertical.

23. A bent tube, in the form of a semicircle, is fixed in a vertical plane with its vertex downwards and its ends in a horizontal line, and is one-third filled with mercury. If one end be closed and the mercury slightly disturbed, prove that the time of an oscillation is $\sqrt{\frac{\pi^2 a}{3g}}$, a being the radius, and the height of the barometer being equal to the length of the tube.

24. Fluid contained in a vessel of infinite extent flows out by a horizontal circular aperture (diameter equal to $2c$) in the base of the vessel: if the axis of x be measured vertically downwards through the centre of the aperture and the axis of y horizontally, shew that the surface of the out-flowing fluid is generated by the revolution about the axis of x of a curve whose differential equation is

$$\frac{d^2y}{dx^2} + \left(\frac{x+k}{a^2} - \frac{hc^4}{a^2y^4}\right)\left(1 + \overline{\frac{dy}{dx}}\Big|^2\right)^{\frac{3}{2}} - \frac{1}{y}\left(1 + \overline{\frac{dy}{dx}}\Big|^2\right) = 0,$$

where a, k and h are constants.

25. A tube in the form of an ellipse is placed with one axis vertical and filled with fluid as far as the extremities of the other axis; the bore of the tube at any point is $\beta + \mu s$, where β and μ are small, and s is the distance of the point from the lowest point of the ellipse measured along the tube. If the fluid be slightly displaced, find the time of a small oscillation; and prove that the length of the simple equivalent pendulum is $\dfrac{\beta + \mu l}{\mu} \log \dfrac{\beta + \mu l}{\beta}$, $4l$ being the circumference of the ellipse.

26. Jets of water escape horizontally from orifices along a generating line of a vertical cylinder kept always full. Shew that (to axes inclined 45° to the vertical) the equation of the lines of equal action for unit mass of water is of the form

$$x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}.$$

Shew also that the line of equal time for particles of water issuing simultaneously from the orifices is the free path of the water which leaves the vessel by an orifice at a depth below the surface due to that time.

CHAPTER XII.

STREAM LINES. KINETIC ENERGY OF A LIQUID. VORTEX MOTIONS.

163. *Lines of Motion, or Stream Lines.*

The direction of the motion of the fluid particle at the point (x, y, z) is defined by the quantities u, v, w, and therefore the differential equations of the stream lines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

These lines intersect at right angles the surfaces of which the differential equation is

$$udx + vdy + wdz = 0,$$

and therefore, if there be a velocity-function, ϕ, the surfaces,

$$\phi = C,$$

are orthogonal to the stream lines.

In general the condition for the existence of such orthogonal surfaces is

$$u\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + v\left(\frac{du}{dz} - \frac{dw}{dx}\right) + w\left(\frac{dv}{dx} - \frac{du}{dy}\right) = 0,$$

a condition which is obviously satisfied in all cases of irrotational motion.

Consider the case of *the steady motion, when irrotational, of a liquid in two dimensions.*

From the given condition, we have

$$\frac{du}{dy} = \frac{dv}{dx} \quad\text{..........................}\quad (\alpha),$$

and, from the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad\text{......................}\quad (\beta).$$

The differential equation of the stream lines is

$$vdx - udy = 0,$$

which, by the equation (β), is a perfect differential.

Let $\psi = C$ be the integral, that is,

let $$u = \frac{d\psi}{dy}, \quad v = -\frac{d\psi}{dx},$$

$$\therefore \text{ from } (\alpha), \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots (\gamma).$$

If a possible value of ψ be found, and the values of u and v be obtained, the pressure is given by the equation,

$$\frac{p}{\rho} = S - \frac{1}{2}(u^2 + v^2) + C,$$

where C is constant along any particular stream line.

Suppose, if possible, that the stream lines are similar concentric conics, given by the equation

$$ax^2 + bxy + cy^2 = d,$$

then we must have, from (γ),

$$a + c = 0,$$

and the conics must be equilateral hyperbolas.

Taking then $\psi = \mu(x^2 - y^2)$, we have

$$u = 2\mu y, \quad v = 2\mu x,$$

and the velocity varies as the distance from the origin.

164. If $F(x + y\sqrt{-1}) = \phi(x, y) + \sqrt{-1}\,\psi(x, y)$, the functions ϕ and ψ are called *Conjugate Functions*, and each of them satisfies the equation (γ).

If ϕ be the *velocity function*, then ψ is called the *current function*, and is such that the difference of its values at two points represents the flow across any line joining the points*.

* For, if ds be an element of a curve, and θ the inclination of the tangent to the axis of x, the flow across the curve

$$= \int (u \sin\theta - v\cos\theta)\, ds = \int \left(\frac{d\psi}{dy}dy + \frac{d\psi}{dx}dx\right) = \int d\psi = \psi_2 - \psi_1.$$

From above we have

$$\frac{d\phi}{dx} = \frac{d\psi}{dy}, \quad \frac{d\phi}{dy} = -\frac{d\psi}{dx};$$

$$\therefore \frac{d\phi}{dx}\frac{d\phi}{dy} + \frac{d\psi}{dx}\frac{d\psi}{dy} = 0,$$

and
$$\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 = \left(\frac{d\psi}{dx}\right)^2 + \left(\frac{d\psi}{dy}\right)^2.$$

Hence the curves obtained by putting ϕ and ψ equal to a constant form an orthogonal system; and, if one denote the velocity function, the other will denote the current function.

From this it follows that if the velocity at any point in a given motion in a plane be turned through a right angle without altering its magnitude, we shall obtain a possible motion, the boundary conditions being suitably altered.

Thus, if $F(x + y\sqrt{-1}) = \log(x + y\sqrt{-1}) = \log(r\epsilon^{\theta\sqrt{-1}})$,

$$= \log r + \sqrt{-1}\,\theta,$$

$$\phi = \log r, \text{ and } \psi = \theta,$$

and we obtain, by taking ϕ as the velocity function, motion diverging uniformly from a centre, and by taking ψ as the velocity function, we obtain vortex motion round the centre, the velocity in each case being inversely as the distance from the centre.

By means of these conjugate functions of a function of a complex, $x + y\sqrt{-1}$, we can obtain an infinite number of possible motions, each defined by a function of the complex.

Thus, if $F(x + y\sqrt{-1}) = \mu(x + y\sqrt{-1})^2$, we fall upon the case of the preceding article; and taking $\mu(x + y\sqrt{-1})^3$, we obtain the system,

$$\phi = \mu x(x^2 - 3y^2), \quad \psi = \mu y(3x^2 - y^2).$$

If we take the function $\sin^{-1}(x + y\sqrt{-1})$, we obtain a system of confocal ellipses and hyperbolas.

165. In the case of the steady motion, *when rotational*, of a liquid in two dimensions, the equations of motion are, Art. 147,

$$-\frac{dQ}{dx}=u\frac{du}{dx}+v\frac{du}{dy},$$

$$-\frac{dQ}{dy}=u\frac{dv}{dx}+v\frac{dv}{dy},$$

from which we obtain

$$\frac{d}{dy}\left(u\frac{du}{dx}+v\frac{du}{dy}\right)=\frac{d}{dx}\left(u\frac{dv}{dx}+v\frac{dv}{dy}\right).$$

As before, the equation of continuity proves that $vdx-udy$ is a perfect differential, $d\psi$, and therefore

$$u=\frac{d\psi}{dy},\quad v=-\frac{d\psi}{dx},\quad\text{and}\quad \frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=-2\zeta,$$

and, by substitution in the preceding equation, we shall obtain the general equation which ψ must satisfy,

$$\frac{d\psi}{dy}\frac{d}{dx}\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right)=\frac{d\psi}{dx}\frac{d}{dy}\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right).$$

This may be written, $u\dfrac{d\zeta}{dx}+v\dfrac{d\zeta}{dy}=0$, which is equivalent to $\dfrac{D\zeta}{dt}=0$.

166. *Stream lines caused by a sphere traversing, in a straight line, an indefinitely large mass of liquid.*

We assume that the sphere and the liquid are originally at rest, so that the motion is irrotational.

This question can be treated in two ways; either directly, or by supposing the whole motion reversed, so that the sphere remains at rest, and the liquid then has, in the opposite direction, the velocity and acceleration of the sphere. We shall first take the latter method, so as to obtain the *relative* stream lines.

If ϕ be the velocity function, the equation of continuity is

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right)=0^*.$$

To obtain a solution of this equation, let $\phi=R\cos\theta$, R being a function of r, then

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-2R=0,$$

and therefore $$\frac{d^2R}{dr^2}+2\left(\frac{1}{r}\frac{dR}{dr}-\frac{R}{r^2}\right)=0,$$

$$\frac{dR}{dr}+\frac{2R}{r}=3A;$$

hence $$r^2\frac{dR}{dr}+2rR=3Ar^2, \text{ and } Rr^2=Ar^3+B;$$

$$\therefore\ R=Ar+\frac{B}{r^2},$$

$$\text{and } \phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta.$$

Let V be the velocity of the sphere, and c its radius; then, when $r=c$, the radial velocity is zero, and, at an infinite distance, it is $-V\cos\theta$.

$$\text{The radial velocity}=\frac{d\phi}{dr}=\left(A-\frac{2B}{r^3}\right)\cos\theta;$$

$$\therefore A=-V,\quad B=-\frac{1}{2}c^3V,$$

$$\text{and } \phi=-V\cos\theta\left\{r+\frac{c^3}{2r^2}\right\}.$$

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* This equation can of course be obtained by transformation from the equation in rectangular co-ordinates, but the simplest method is to express the fact that a small polar element of space remains full of liquid.

Employing ω for the azimuthal angle this gives

$$\delta r\frac{d}{dr}\left\{\frac{d\phi}{dr}r^2\sin\theta\,\delta\theta\,\delta\omega\right\}+\delta\theta\frac{d}{d\theta}\left\{\frac{d\phi}{rd\theta}r\sin\theta\,\delta r\,\delta\omega\right\}=0,$$

from which the equation above given follows at once.

At the point (r, θ),

$$\left.\begin{aligned}&\text{the radial velocity} &&= -V\cos\theta\left(1-\frac{c^3}{r^3}\right)\\ &\text{the transversal velocity} &&= \frac{d\phi}{rd\theta} = V\sin\theta\left(1+\frac{c^3}{2r^3}\right)\end{aligned}\right\}\ldots\ldots\ldots (\alpha),$$

and the equation of the lines of motion is

$$\frac{dr}{\left(\frac{c^3}{r^3}-1\right)\cos\theta} = \frac{rd\theta}{\left(1+\frac{c^3}{2r^3}\right)\sin\theta},$$

or
$$-2\cot\theta d\theta = \frac{2r^3+c^3}{r^3-c^3}\frac{dr}{r} = \left(\frac{3r^2}{r^3-c^3}-\frac{1}{r}\right)dr;$$

$$\therefore\ \sin^2\theta = \frac{Cr}{r^3-c^3},$$

an equation which defines the stream lines relative to the sphere.

To find the resultant pressure on the sphere, we have, if q be the velocity of the liquid, and f the acceleration of the sphere,

$$\begin{aligned}\frac{p}{\rho} &= F(t) - fx - \frac{d\phi}{dt} - \frac{1}{2}q^2\\ &= F(t) - \frac{1}{2}q^2 - fx + \left(r+\frac{c^3}{2r^2}\right)f\cos\theta\\ &= F(t) - \frac{1}{2}q^2 + \frac{c^3}{2r^2}f\cos\theta;\end{aligned}$$

when $r = c$, the velocity is wholly transversal and, from the equations (α), is equal to $\frac{3}{2}V\sin\theta$;

$$\therefore\ \frac{p}{\rho} = F(t) - \frac{9}{8}V^2\sin^2\theta + \frac{1}{2}f\cos\theta,$$

and the resultant pressure on the sphere

$$= \int_{-c}^{c} p\,.\,2\pi c dx\,.\cos\theta = \int_{-c}^{c} 2\pi p x dx.$$

The part of the summation due to the first two terms of p disappears, and we find that, if M be the mass of the sphere,

the pressure $$= \frac{2}{3}\pi\rho f c^3 = \frac{1}{2} Mf.$$

From this result it follows that if the sphere move with constant velocity, there is no resistance*.

167. In solving the same question directly, we take for the origin the instantaneous position in space of the centre of the sphere. The velocity-function is of the same form as before, but we must now observe that, at infinity, $\frac{d\phi}{dr} = 0$, and, if $r = c$,

$$\frac{d\phi}{dr} = V\cos\theta; \quad \therefore \phi = -\frac{c^3 V\cos\theta}{2r^2}.$$

To find the *actual* lines of motion, we have

$$\frac{dr}{\frac{c^3 V\cos\theta}{r^3}} = \frac{r\,d\theta}{\frac{c^3 V\sin\theta}{2r^3}},$$

and $\therefore r = C\sin^2\theta$, and the relative stream lines can be deduced from this result.

According to this theory the pressure on the sphere vanishes when the velocity is constant, a result entirely contradicted by experience.

The discrepancy arises from our having dealt with the imaginary case of a perfect liquid; in any actual experiment there will be frictional action of the fluid particles on each other and on the surface of the sphere.

168. *Stream lines caused by a circular cylinder of infinite length traversing, in a direction perpendicular to its axis, an indefinitely extended mass of liquid.*

The equation of continuity is

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \frac{d^2\phi}{d\theta^2} = 0,$$

* Mr Froude, in the papers in *Nature* referred to on page 216, considers that the surface-friction, or skin-friction, constitutes almost the whole of the resistance experienced by bodies of tolerably easy shape travelling under water at any reasonable speed.

and, if $\phi = R\cos\theta$,

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - R = 0.$$

Writing this equation in the form

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \left(r\frac{dR}{dr} + R\right) = 0,$$

we obtain

$$r^2\frac{dR}{dr} - Rr = -2B,$$

or

$$\frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2} = -\frac{2B}{r^3};$$

$$\therefore\ R = Ar + \frac{B}{r}.$$

Reversing the motion of the cylinder, so as to obtain the relative stream lines, the conditions are

$$\frac{d\phi}{dr} = -V\cos\theta, \text{ when } r = \infty,$$

$$\frac{d\phi}{dr} = 0, \text{ when } r = c.$$

Hence we obtain

$$\phi = -V\cos\theta\left(r + \frac{c^2}{r}\right),$$

and the equation to the relative stream lines is

$$\frac{dr}{-\cos\theta\left(1 - \dfrac{c^2}{r^2}\right)} = \frac{r\,d\theta}{\dfrac{\sin\theta}{r}\left(r + \dfrac{c^2}{r}\right)},$$

or

$$\frac{r^2 + c^2}{r^2 - c^2}\frac{dr}{r} = -\cot\theta\,d\theta.$$

Integrating, we obtain the equation

$$\left(r - \frac{c^2}{r}\right)\sin\theta = C.$$

If the question be solved directly, that is, without reversing the motion of the cylinder, we shall find that

$$\phi = -\frac{c^2}{r} V \cos\theta,$$

and that the actual stream lines are circles,

$$r = C \sin\theta.$$

Proceeding as in the case of the sphere, we find that if M be the mass of the cylinder, the resultant pressure on the cylinder is equal to Mf, which vanishes when the velocity is constant.

Mr Ferrers, in the volumes for 1874 and 1875 of the *Mathematical Journal*, has worked out the cases of an elliptic cylinder, and of an ellipsoid, traversing an infinite mass of liquid.

Vortex Motions.

169. In general the motion of a small element of fluid is compounded of three separate changes; namely, a translation in space, a dilatation or contraction, and a circulation about some instantaneous axis.

The term *Vortex Motion* is applied by Helmholtz to all cases of rotational motion.

Vortex-lines are lines drawn through the fluid so as to coincide, at every point, with the direction of the instantaneous axis of circulation at that point.

Vortex-filaments are the portions of fluid bounded by the series of vortex-lines drawn through every point of the perimeter of a very small closed curve.

We assume that the forces in action are derivable from a potential, and we confine our attention to the case of a homogeneous liquid. The two following theorems are fundamental properties of vortex-lines.

170. *Any vortex-line, however it may be translated, is composed of the same elements.*

The initial equations of a vortex-line are

$$\frac{da}{\xi_0} = \frac{db}{\eta_0} = \frac{dc}{\zeta_0} = \lambda.$$

Now, x, y, z being the co-ordinates at any time of the particle originally at a, b, c,

$$dx = \frac{dx}{da}da + \frac{dx}{db}db + \frac{dx}{dc}dc$$

$$= \lambda\left(\xi_0\frac{dx}{da} + \eta_0\frac{dx}{db} + \zeta_0\frac{dx}{dc}\right)$$

$$= \lambda\xi \text{ (Art. 145)};$$

$$\therefore \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \lambda;$$

that is, the vortex-line is composed of the same elements as at first.

Observing that, if the motion of any portion of liquid be once irrotational it is always so, it is clear that no additions will be made to the quantity of liquid in the vortex-filaments.

171. *The product of the angular velocity of any element of a very thin vortex-filament, and of its cross section at that point, is at all times constant, and is the same throughout its whole length.*

If ds be an element of the length of the filament and ω the resultant angular velocity,

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{ds}{\omega};$$

and, from the preceding Article, each of these ratios

$$= \lambda = \frac{ds_0}{\omega_0},$$

and therefore $$\frac{ds}{\omega} = \frac{ds_0}{\omega_0}.$$

But the volume of the element of the vortex-filament

$$= \alpha ds = \alpha_0 ds_0,$$

if α be the areal section;

$$\therefore \alpha\omega = \alpha_0\omega_0,$$

that is, $\alpha\omega$ is independent of the time.

For the second part of the theorem, observe that

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0,$$

and therefore $$\iiint \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) dxdydz = 0,$$

the integration extending over any portion of the liquid.

Let l, m, n be the direction-cosines of the normal to the bounding surface, drawn outwards, at any point;
then, integrating with respect to x, y, and z respectively, we have

$$\iint \xi dydz + \iint \eta dzdx + \iint \zeta dxdy = 0,$$

and therefore $$\iint (l\xi + m\eta + n\zeta)\, dS = 0,$$

the integration extending over the surface of which dS is an element.

If θ be the angle between the vortex-line and the normal, the preceding equation becomes

$$\iint \omega \cos\theta dS = 0.$$

Apply this to the portion of a vortex-filament bounded by two plane ends perpendicular to its axis, of which the areas are α_1 and α_2, and observe that $\cos\theta = 0$, except at the ends, and $= 1$ at one end, and $= -1$ at the other;

hence, if ω_1, ω_2 be the angular velocities at the ends,

$$\omega_1\alpha_1 = \omega_2\alpha_2,$$

that is, $\omega\alpha$ is uniform throughout the length of the filament.

Hence it follows that a vortex-filament must either end at the free surface or at any other boundary of the liquid, or else must return into itself, forming a closed ring. For if a filament

be imagined to end in the liquid, a closed surface could be found for which the expression $\iint \omega \cos\theta \, dS$ would not vanish.

172. In order to examine a simple case, imagine the existence of straight parallel vortex-filaments, either in an indefinitely extended mass of liquid, or in a mass limited by two planes perpendicular to the filaments.

Taking the axis of z parallel to the filaments, we have

$$w = 0, \quad \frac{du}{dz} = 0, \quad \text{and} \quad \frac{dv}{dz} = 0,$$

and therefore

$$\xi = 0, \quad \eta = 0, \quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy}.$$

Also the equation

$$\frac{D\zeta}{dt} = \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\frac{dw}{dz}, \text{ Art. 147,}$$

proves that ζ is constant.

The equation of the lines of motion is

$$vdx - udy = 0,$$

and it follows from the equation of continuity that $vdx - udy$ is a perfect differential $d\psi$;
hence, as before,

$$u = \frac{d\psi}{dy}, \quad v = -\frac{d\psi}{dx},$$

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = -2\zeta,$$

and the lines of motion are given by the equation $\psi = C$.

From the form of the equation it follows that ψ is the potential at any point of an infinite medium, the density of which is zero, except along the vortex-filaments, which may be looked upon as physical straight lines of density $\frac{\zeta}{2\pi}$.

Hence its differential coefficients are the resultants of the attractions of these lines.

173. Supposing that only a single vortex-filament is in existence, at the point (a, b), and that α is its areal section, we have

$$v = -\frac{d\psi}{dx} = \frac{2\alpha}{r}\left(\frac{\zeta}{2\pi}\right)\frac{x-a}{r} = \frac{\zeta\alpha}{\pi}\frac{x-a}{r^2},$$

and
$$u = \frac{d\psi}{dy} = -\frac{2\alpha}{r}\left(\frac{\zeta}{2\pi}\right)\frac{y-b}{r} = -\frac{\zeta\alpha}{\pi}\frac{y-b}{r^2},$$

r being the distance between the points (a, b) and (x, y).

Hence, if q be the resultant velocity,

$$q = \frac{\zeta\alpha}{\pi r},$$

and the direction of q is perpendicular to the distance r.

174. If there be any number of vortex-filaments, the velocity at any point will be determined by the superposition of the velocities due to each, and will be expressed by the equations,

$$v = \Sigma\frac{\zeta\alpha}{\pi}\frac{x-a}{r^2}, \quad u = -\Sigma\frac{\zeta\alpha}{\pi}\frac{y-b}{r^2}.$$

In the case of any number of filaments, if we denote by m the expression $\zeta\alpha$, which is called the *strength* of the vortex, and if u, v be the component velocities of a filament, the expressions

$$\Sigma(mu) \text{ and } \Sigma(mv)$$

will both vanish, for they consist of pairs of terms of the forms

$$m_1\frac{m_2}{\pi}\frac{x_1-x_2}{r^2} \text{ and } m_2\frac{m_1}{\pi}\frac{x_2-x_1}{r^2}.$$

Hence, regarding m as a mass, the centre of gravity of the vortex-filaments remains stationary during their motions about one another.

Consider the case of two vortex-filaments of small section. Each will produce a motion on the other perpendicular to the line joining them, and they will turn about their common centre of gravity at a constant distance from each other.

If the rotations are in contrary directions, and if moreover the strengths of the two filaments are equal, their centre of gravity will be at an infinite distance, and they will move in parallel directions with equal velocities.

Now consider the case of a single vortex-filament moving near an infinite plane to which it is parallel.

The liquid close to the plane can only move parallel to the plane, and this condition will be fulfilled if we remove the plane, and imagine in its place an infinite mass of fluid with another filament, the image with respect to the plane of the first.

The filament will clearly move parallel to the plane.

This discussion of the case of straight filaments is taken, with slight alterations, from Professor Tait's translation of Helmholtz's Paper.

For an extended study of vortex motions, the student is referred to Helmholtz's Paper, to Kirchhoff's *Vorlesungen über Mathematische Physik*, and to Sir William Thomson's paper in the Transactions, for the year 1869, of the Royal Society of Edinburgh.

175. *The Kinetic Energy of a mass of homogeneous liquid in a state of irrotational motion.*

If K represent the kinetic energy,

$$\frac{2K}{\rho} = \iiint (u^2 + v^2 + w^2)\, dxdydz$$

$$= \iiint \left(u\frac{d\phi}{dx} + v\frac{d\phi}{dy} + w\frac{d\phi}{dz}\right) dxdydz.$$

Integrating by parts,

$$\frac{2K}{\rho} = \iint \phi(udydz + vdzdx + wdxdy)$$

$$- \iiint \phi\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dxdydz,$$

the double integration extending over the bounding surface.

The triple integral vanishes by the equation of continuity; and if dS be an element of the surface and λ, μ, ν the direction-cosines of the normal drawn outwards,

$$\frac{2K}{\rho} = \iint \phi(\lambda u + \mu v + \nu w) dS.$$

If dn be an element of the normal at dS drawn outwards,

$$\lambda u + \mu v + \nu w = \frac{d\phi}{dn},$$

$$\therefore K = \frac{1}{2}\rho \iint \phi \frac{d\phi}{dn} dS \text{ ....} (a).$$

It should be noticed that we have the condition,

$$\iint \frac{d\phi}{dn} dS = 0,$$

expressing the fact that the volume is constant.

176. If ϕ be a single-valued function, the equation

$$\iiint (u^2 + v^2 + w^2)\, dxdydz = \iint \phi \frac{d\phi}{dn} dS,$$

which, it will be observed, is a particular case of Green's Theorem, is true in all cases, whether the closed space filled by the liquid is simply-connected, or multiply-connected*.

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* In a simply-connected space any two lines joining two given points may be so varied as to coincide with each other without leaving the space in question.

In a doubly-connected space two lines may be so drawn that they cannot be made to coincide with each other without one or the other passing out of the space.

Such lines are said to be irreconcilable with each other.

If two lines be drawn joining two points in a space, whether simply connected or doubly connected, which can be made to coincide without passing out of the space, they are said to be mutually reconcilable.

Thus, the inside of a solid ring is a doubly-connected space.

In an n-ly connected space, the bounding surface is such that n irreconcilable lines can be drawn between any two given points within the space.

It is possible, however, that ϕ may be a many-valued function, and that the velocity may be infinite within the liquid, in which case the equation is not true.

Suppose for instance that

$$u = \frac{-c^2 y}{x^2+y^2} \text{ and } v = \frac{c^2 x}{x^2+y^2}.$$

These expressions are derivable from the velocity-function

$$\phi = c^2 \tan^{-1}\frac{y}{x},$$

which has an infinite number of values.

The equation of continuity is satisfied, the expression

$$\frac{dv}{dx} - \frac{du}{dy}$$

vanishes except along the axis of z, and the resultant velocity

$$= \sqrt{u^2+v^2} = \frac{c^2}{\sqrt{x^2+y^2}} = \frac{c^2}{r}.$$

The motion is therefore irrotational, except along the axis of z, which is a vortex-line.

This is Rankine's Free Circular Vortex.

177. From the equation (a) it follows that if a rigid closed vessel at rest be filled with liquid, and if the interior of the vessel be a simply-connected space, there can be no irrotational motion of the liquid.

For, if $\frac{d\phi}{dn} = 0$, then u, v, and w are necessarily zero.

178. THEOREM. *If the bounding surface of a liquid, originally at rest, be made to vary in a given arbitrary manner, the kinetic energy of the liquid at each instant will be less than it would be if the liquid had any other motion consistent with the given motion of the bounding surface.*

Let u_1, v_1, w_1 be the components of the velocity at the point x, y, z in any other possible state of motion;

then $$lu_1 + mv_1 + nw_1 = H, \text{ at the surface, } \quad\text{.........}(a)$$

and $$\frac{du_1}{dx}+\frac{dv_1}{dy}+\frac{dw_1}{dz}=0,$$

throughout the mass, and u_1, v_1, w_1 may be any quantities satisfying those conditions.

Let K_1 represent the hypothetical kinetic energy; then, taking the density to be unity,

$$2K_1-2K=\iiint(u_1^2-u^2+v_1^2-v^2+w_1^2-w^2)\,dxdydz$$
$$=\iiint\{(u_1-u)^2+(v_1-v)^2+(w_1-w)^2+2u(u_1-u)$$
$$+2v(v_1-v)+2w(w_1-w)\}\,dxdydz.$$

The liquid being originally at rest, the motion is irrotational,

and $\therefore \displaystyle\iiint\left\{u(u_1-u)+v(v_1-v)+w(w_1-w)\right\}dxdydz$

$$=\iiint\left\{\frac{d\phi}{dx}(u_1-u)+\frac{d\phi}{dy}(v_1-v)+\frac{d\phi}{dz}(w_1-w)\right\}dxdydz$$

$$=\iint\phi\{(u_1-u)\,dydz+(v_1-v)\,dzdy+(w_1-w)\,dzdy\}$$

$$-\iiint\phi\left\{\frac{d}{dx}(u_1-u)+\frac{d}{dy}(v_1-v)+\frac{d}{dz}(w_1-w)\right\}dxdydz.$$

The double integral

$$=\iint\phi\{(u_1-u)l+(v_1-v)m+(w_1-w)n\}dS,$$

which vanishes by the equation (a), and the triple integral vanishes by the equation of continuity; and therefore we have

$$2K_1-2K=\iiint\{(u_1-u)^2+(v_1-v^2)+(w_1-w)\}^2dxdydz,$$

an expression essentially positive, unless

$$u_1=w,\ v_1=v, \text{ and } w_1=w.$$

The preceding theorem is given by Sir William Thomson in the volume for the year 1849 of the *Cambridge and Dublin Mathematical Journal.*

179. *Flow and Circulation.* The line-integral of the tangential velocity along any line, lying entirely within the fluid, is called the flow along that line.

If the line form a closed curve, and the integral be taken over the whole of the curve, the flow is called the circulation in that curve.

$$\text{Thus the flow} = \int\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right)d\phi$$

$$= \int(udx + vdy + wdz).$$

If the motion be irrotational, the flow $= \int d\phi = \phi_2 - \phi_1$, where ϕ_2 and ϕ_1 are the values of ϕ at the two ends of the line in question, and it is assumed that ϕ is a single-valued function.

Hence it follows that in the case of irrotational motion, if two points in the fluid be taken, the flow is, at any instant, the same along any line joining the two points, provided the line be entirely within the fluid, and the velocity-function single-valued.

This assumes that the lines along which the flow is taken are mutually reconcilable, or that each line can be changed into any other without passing out of the space for which the values of $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$ and $\frac{d\phi}{dz}$ are finite.

Under the same restrictions the circulation is zero round every closed curve.

In this case the curve must be capable of being contracted to a point without passing out of the space through which the differential coefficients of ϕ have finite values.

Thus, if $\phi = c^2 \tan^{-1}\frac{y}{x}$, the flow is the same for any two lines joining two given points, provided the axis of z is not between the lines; and the circulation is zero round any closed curve which does not include the axis of z within its perimeter.

If, however, we take a circle having its centre on the axis of z, the circulation round this circle is the change in the value of ϕ in passing round, which is $2\pi c^2$.

180. Assuming the existence of a Force Potential the equations of motion are, (Art. 147),

$$\frac{Du}{dt} = -\frac{dQ}{dx}, \quad \frac{Dv}{dt} = -\frac{dQ}{dy}, \quad \frac{Dw}{dt} = -\frac{dQ}{dz}.$$

If ζ be the tangential component of the velocity at any point of a curve joining two assigned particles P_1, P_2 of the fluid, and always passing through the same particles,

$$\zeta ds = udx + vdy + wdz;$$

$$\therefore \frac{D}{dt}(\zeta ds) = \frac{Du}{dt}dx + u\frac{D}{dt}dx + \&c.$$

$$= \frac{Du}{dt}dx + udu + \&c. = \frac{1}{2}dq^2 - dQ,$$

q being the resultant velocity.

If $U = \int \zeta ds$, so that U represents the flow from P_1 to P_2,

$$\frac{DU}{dt} = \left\{\frac{1}{2}q^2 - Q\right\}_2 - \left\{\frac{1}{2}q^2 - Q\right\}_1.$$

Hence it follows that *the circulation in any closed line moving with the fluid remains constant through all time.*

We hence obtain another proof of the fact that the motion of a fluid, if once irrotational, is always irrotational.

For if at any instant the motion is irrotational, the circulation round any closed curve is zero (Art. 179), and therefore by the preceding theorem it is always zero, and hence it follows that a velocity-function always exists, that is, the motion is always irrotational.

181. If a solid body rotate about an axis, the circulation round any closed curve in a plane perpendicular to the axis and carried with the body, is equal to twice the area enclosed multiplied by the angular velocity. For if r, θ be co-ordinates of

a point in the curve, r being measured from the axis of rotation,

$$\zeta = r\omega \,.\, r\frac{d\theta}{ds},$$

and therefore $\int \zeta ds = \int \omega r^2 \, d\theta = 2\omega \,.\, (\text{the area}).$

By analogy with this case, if a fluid move in any manner, the circulation round any infinitesimal plane area, divided by twice the area, is the definition of the component rotation of the fluid about an axis perpendicular to the plane of the fluid within the area.

We can apply this definition to find the angular velocities at any point.

For the circulation round the element $dydz$ in the plane yz

$$= vdy - wdz - \left(v + \frac{dv}{dz}dz\right)dy + \left(w + \frac{dw}{dy}dy\right)dz = \left(\frac{dw}{dy} - \frac{dv}{dz}\right)dydz,$$

and therefore $$\xi = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),$$

and similarly we obtain

$$\eta = \frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \quad \zeta = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right).$$

If any open finite surface, lying within a fluid, be divided into parts in any way, it is obvious that the circulation round the boundary of the surface is equal to the sum of the circulations round the boundaries of the parts.

Hence it follows that, if l, m, n be direction cosines of the normal at any point of an element dS of the surface,

$$\iint dS\left\{l\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + m\left(\frac{du}{dz} - \frac{dw}{dx}\right) + n\left(\frac{dv}{dx} - \frac{du}{dy}\right)\right\}$$
$$= \int (udx + vdy + wdz),$$

the surface integral extending over the whole of the surface, and the line integral round its boundary.

The theorems of the three preceding Articles are taken from Sir William Thomson's paper on Vortex Motion in the *Edinburgh Transactions* for 1869.

EXAMPLES.

1. Prove that for irrotational motion in two dimensions, the differential equation of the stream lines in polar co-ordinates is

$$r^2 \frac{d^2\psi}{dr^2} + r\frac{d\psi}{dr} + \frac{d^2\psi}{d\theta^2} = 0.$$

Shew hence that a system of confocal and coaxial parabolas is a possible system of stream-lines.

2. In the case of irrotational motion of a liquid in two dimensions, shew that the velocity, V, satisfies the equation,

$$\log V = F(x + y\sqrt{-1}) + f(x - y\sqrt{-1}).$$

In the same case, if a very small elliptic portion of the liquid, having its axes parallel to the co-ordinate axes, be suddenly solidified, and the liquid around it annihilated at the same instant, prove that the ellipse will rotate with the angular velocity $\frac{e^2}{2-e^2}\frac{du}{dy}$, where e is the eccentricity.

3. If for homogeneous liquid moving in one plane there exist a velocity-function of the form $x^2 f\left(\frac{y}{x}\right)$, find the velocity at any point, and the path of a particle of the liquid.

4. A closed vessel is filled with water which is at rest, and the vessel is then moved in any manner; apply the principle of the conservation of areas to prove that, if the vessel have any motion of rotation, no finite portion of the water can remain at rest relatively to the vessel.

5. A mass of homogeneous liquid subject to no external force but gravity is in motion; prove that the particles cannot describe circles about a common vertical axis unless the surfaces of equal velocity be cylinders, and find an expression for the angular velocity of each cylinder if all surfaces of equal pressure are spheres.

6. If $u = \dfrac{ax - by}{x^2 + y^2}$, $v = \dfrac{ay + bx}{x^2 + y^2}$, and $w = 0$, describe the nature of the motion of the liquid. (Rankine's *Free Spiral Vortex.*)

7. Describe the motion of a liquid when the velocity-function is

$$\omega c\,(y \cos \omega t - x \sin \omega t).$$

8. In the case of motion in two dimensions for which

$$udx + vdy = \tfrac{1}{2}\, d\,(y^2 - x^2),$$

pply the differential equation of the bounding surface to obtain the eneral equation of lines made up of the same particles; and thence hew that the particles which once lie in a curve of the n^{th} order coninue to lie in a curve of the n^{th} order.

9. A homogeneous liquid, enclosed in a boundary which can hange both in shape and area, but not in volume enclosed, is acted n by a force whose components are

$$y + z + \frac{k}{x + y + z},\quad z + x + \frac{k}{x + y + z},\quad x + y + \frac{k}{x + y + z},$$

espectively; when the time $t = 0$, the liquid is at rest, and the presure $= k\rho \log \dfrac{x + y + z}{h}$; afterwards the pressure at the boundary is lways

$$k\rho \log \frac{x + y + z}{h} - \rho t^2 (x^2 + y^2 + z^2 + xy + yz + zx) - \rho F(t):$$

rove that the components of the velocity will always be $t(y + z)$, $(z + x)$, $t(x + y)$, and that the curve described by the particle, whose o-ordinates, when $t = 0$, were (x_0, y_0, z_0), has for its equations

$$\left(\frac{x - y}{x_0 - y_0}\right)^2 = \left(\frac{y - z}{y_0 - z_0}\right)^2 = \frac{x_0 + y_0 + z_0}{x + y + z}.$$

10. A mass M of liquid is running round a circular channel f radius a with velocity u: another equal mass of liquid is running ound a channel of radius b with velocity v; the radius of the one hannel is made to increase and the other to diminish till each has he original value of the other: shew that the work required to prouce the change is

$$\frac{1}{2}\left(\frac{v^2}{a^2} - \frac{u^2}{b^2}\right)(b^2 - a^2) M.$$

Hence shew that the motion of a liquid in a circular whirlpool

will be stable or unstable according as the areas described by particles in equal times increase or diminish from centre to circumference.

11. A given quantity of liquid moves in a smooth conical tube having a small vertical angle, and the distances of its nearer and farther extremities from the vertex at the time t are r and r'; shew that

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left\{3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right\}=0,$$

the pressures at the two surfaces being equal.

Shew also that the preceding equation results from supposing the vis viva of the mass of liquid to be constant; and that the velocity of the inner surface is given by the equations

$$V^2=\frac{Cr'}{r^3(r'-r)},\quad r'^3-r^3=c^3,$$

C and c being constants.

12. In the irrotational motion of a liquid in two dimensions, prove that the motion derived from it by turning the direction of motion at each point in one direction through 90° without changing the velocity will also be a possible irrotational motion, the conditions at the boundaries being altered so as to suit the new motion.

Obtain in this way vortex motion about a straight axis from motion diverging symmetrically from the axis, and write down the velocity-function in each case.

If the axis of the vortex be the axis of z, measured vertically downwards, the plane of (xy) the asymptotic plane to the free surface, and if ϖ be the atmospheric pressure; prove that the equation of the surface at which the pressure is $\varpi+g\rho a$ is

$$(x^2+y^2)(z-a)=c^3,$$

where c is a constant.

13. Every particle of a mass of homogeneous liquid is revolving uniformly about an axis, the angular velocity varying as the n^{th} power of the distance from the axis. A small and spherical portion is suddenly solidified. Shew that it will begin to rotate about an axis through its centre with an angular velocity equal to $\frac{n+2}{2}$ of that with which it is revolving about the fixed axis in the liquid.

14. The base of an infinite cylinder is the space contained between an equilateral hyperbola and its asymptotes. A plane is drawn perpendicular to the base, and cutting it in a straight line parallel to an asymptote, and the portion of the cylinder between this plane and its parallel asymptote is filled with homogeneous liquid, under the action of no impressed forces. The plane being suddenly removed, determine the motion; and prove that the free surface of the liquid will remain plane, and advance with a uniform velocity proportional to $\sqrt{\varpi}$, where ϖ is the pressure at an infinite distance, which is supposed to remain constant throughout the motion.

15. A mass of fluid is in motion in any manner, and an indefinitely small portion, the principal moments of which are A, B, C, is suddenly solidified; prove that, if the coordinate axes coincide in direction with the principal axes of the solidified portion, and u, v, w, be the component velocities at the points (x, y, z), the initial component angular velocities of the solidified portion will respectively be

$$\frac{A+B-C}{2A}\frac{dv}{dz}-\frac{A-B+C}{2A}\frac{dw}{dy},\quad \frac{-A+B+C}{2B}\frac{dw}{dx}-\frac{A+B-C}{2B}\frac{du}{dz},$$

$$\frac{A-B+C}{2C}\frac{du}{dy}-\frac{-A+B+C}{2C}\frac{dv}{dx}.$$

16. The motion of a mass of fluid is referred to three rectangular axes, and a small closed curve is described about the origin in the plane of xy; prove that the mean tangential velocity along that curve is to the mean normal velocity as $\frac{dv}{dx}-\frac{du}{dy}$ is to $\frac{du}{dx}+\frac{dv}{dy}$, and that if the curve be a circle each of them is proportional to its radius.

17. A solid body, rotating with uniform velocity ω about a fixed axis, contains a closed tubular channel of small uniform section filled with a liquid in relative equilibrium; if the rotation of the solid body were suddenly destroyed, the liquid would move in the tube with a velocity $=\frac{2A\omega}{l}$, where A is the area of the projection of the axis of the tube on a plane perpendicular to the axis of rotation, and l is the whole length of the tube.

18. If a thin spherical shell, filled with liquid, move through an

infinite mass of liquid originally at rest, the kinetic energy of the liquid outside is half the kinetic energy of the liquid inside the shell.

19. If within an infinite mass of liquid there be supposed to exist certain origins whence liquid is poured into the surrounding mass, shew that (i) the surfaces $\phi =$ constant, may be closed surfaces, (ii) the kinetic energy in steady motion of the shell of liquid between $\phi = c_1$ and $\phi = c_2$ is $\frac{1}{2}V(c_1 - c_2)$, where V is the volume of liquid poured into the mass from the origins within the surfaces.

20. In a liquid in motion an imaginary surface is drawn; prove that the momentum of the liquid in this surface parallel to the axis of x is $\iint NxdS$, where N is the normal velocity at the surface, x the co-ordinate of any point on the surface, and dS the element of the surface at the point.

21. Shew that

$$\phi = \log . \frac{\sqrt{(x+a)^2 + y^2}}{\sqrt{(x-a)^2 + y^2}}$$

gives a possible motion in two dimensions. Find the form of the stream-lines for this motion; and prove that the curves of equal velocity are lemniscates.

22. Two infinite parallel planes are placed indefinitely near to one another and the space between them is filled with liquid. Through a small aperture at the point A in one of the planes liquid is forced in at a uniform rate and is drawn off at the same rate through an equal aperture at the point B distant $2a$ from A. The motion being steady, if the lines of flow be the system of circles passing through A and B, and the fluid on entering at A *begin* to move with the same velocity in all directions, shew that the velocity at any point P will vary inversely as the product of the radius of the circle through A, B and P, and the distance of P from the line AB. Shew that the motion is irrotational, except at the points A and B, and find the current lines.

CHAPTER XIII.

THEORY OF SOUND.

182. **The** theory of sound is included in the theory of the small oscillations of elastic fluids; that this is the case follows from the consideration of a few experimental facts.

In the first place, the effect on the organs of the ear, called sound, is not produced unless there is an atmospheric communication between the ear and the disturbance causing the sound; if a bell is placed under a receiver, and the receiver exhausted as nearly as possible, the striking of the bell is not heard at all; moreover if the bell be struck during the process of exhaustion, the sound becomes gradually more faint as the exhaustion proceeds: it is evident therefore that the intensity of sound depends upon the density of the air, and diminishes with the diminution of that density.

That there is an actual motion of the atmosphere is shewn by the mechanical action which can sometimes be observed; for instance, glass windows are shaken and are sometimes broken by the firing of cannon; and similar effects may be produced by the sounds of an organ. It is well known that a musical note, sounded on any instrument, may produce a vibration, in unison with it, in some other body with which the instrument is not in contact; the human voice will, for example, set in motion a pianoforte wire, if the note sounded be in unison with the fundamental note of the wire, and this, it is evident, can only be effected by the transmission through the air of a mechanical action. Again, it is observed that, when sounds are heard through an atmosphere loaded with particles of dust, there is no sensible motion of the particles; and, in general, that sound is not necessarily accompanied by *wind*, unless the observer be near the origin of the sound: it

follows from facts of this kind that sound is caused by *small* motions of the aerial fluid.

The Velocity of Sound.

183. It is a matter of very ordinary observation, that sound requires time for its propagation; a person standing near a cannon when it is fired, will hear the report almost at the same instant that the flash is visible to him; if, however, the cannon is at a distance, there will be a sensible interval between his perception of the flash and the report, and this interval increases with the distance.

It has been observed, moreover, that the velocity of sound is increased when the temperature is raised.

A great number of experiments have been made with the view of determining the velocity of sound, but from the various circumstances which affect its propagation there are considerable discrepancies in the results obtained; Sir John Herschel considers that in dry air, at freezing temperature, the best approximation to the velocity of sound is about 1089 feet per second.

From experiments made by Arago and others in 1822, the velocity of sound when the barometer was at 29.8 inches, and the thermometer at 61°, was found to be 1118.4 feet per second.

184. *Sounds of different pitch and intensity travel with the same velocity.*

When a musical band is heard at a distance the harmony is unaffected, and it is therefore clear that there is no sensible difference in the periods of time required for the transit of the various notes produced at the same instant. This inference, however, can only be drawn for the limits of distance within which it is possible to hear the band at all; and it does not appear that direct experiments have been made for a greater distance than 951 metres, or about 1040 yards*.

* For an account of these various experiments, see Herschel's *Sound, Encyc. Metrop.* A piece of evidence may here be given, with reference to Art. (176). On a fine and still evening of June, 1858, the *Messiah* was performed in a tent, and the Hallelujah Chorus was distinctly heard, without loss of harmony, at a distance of two miles.

185. As it is through the air that sound is transmitted to the senses, the especial problem which offers itself, is the discussion, under various conditions, of the small vibrations of the aerial particles; but for a full consideration of the question, the laws of vibration of strings, of elastic rods and plates, of stretched surfaces, and of elastic solids, require to be investigated.

These latter give rise to vibrations of the air, and the determination of the various modes in which their vibrations take place, forms, properly speaking, a part of the general question.

The investigation of the oscillatory movements of a solid body gives rise to equations of considerable complexity, and moreover the most important cases, those of musical sounds, depend in general upon the vibrations of strings, or rods, or of the air in cylindrical tubes; to these cases our attention will be confined.

Effect of Condensation on Temperature.

186. It is an experimental fact that heat is produced by the sudden compression of air, and that, on the other hand, heat is lost by its sudden rarefaction; it follows therefore, in the small vibrations producing sound, in which the compressions and rarefactions take place very rapidly, that the air is rendered more elastic, or less elastic, in a greater degree than is given by Boyle's law.

Taking β to represent the ratio of the specific heat of air at constant pressure to the specific heat at constant volume,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\beta. \quad \text{Art. 93.}$$

If then a small portion of fluid, the density of which is ρ_0, and temperature zero, be suddenly compressed so that its density is $\rho_0(1+s)$, s being a small quantity, we have

$$\frac{p}{p_0} = (1+s)^\beta = 1 + \beta s,$$

neglecting the square of s, and

$$\therefore p = k\rho_0(1+\beta s).$$

187. *A hollow cylinder of indefinite length is filled with*

homogeneous air, a portion of which is disturbed in such a manner that all the particles in any section, perpendicular to the axis, are under the same initial circumstances of displacement;— it is required to determine the resulting motion.

Let ρ_0 be the density of the air when undisturbed and $k\rho_0$ its pressure. At the time t and at distance x measured parallel to the axis, let u be the velocity, p the pressure, and ρ the density.

Neglecting the action of gravity and supposing the surface of the cylinder perfectly smooth, the equation of motion is

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{Du}{Dt},$$

$$\text{or } \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} - u\frac{du}{dx}.$$

If $\quad \rho = \rho_0(1+s), \quad p = k\rho_0(1+\beta s),$

$$\text{and } \frac{dp}{\rho} = \frac{k\beta ds}{1+s},$$

$$\text{or } \frac{k\beta ds}{1+s} = \left(-\frac{du}{dt} - u\frac{du}{dx}\right)dx.$$

If ϕ be a function of t such that $u = \dfrac{d\phi}{dx}$, we obtain, by integration with regard to x,

$$k\beta \log(1+s) = -\frac{d\phi}{dt} - \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2.$$

The motion is supposed to be so small that the square of the velocity may be neglected, and therefore expanding log $(1+s)$,

$$k\beta s = -\frac{d\phi}{dt} \quad \text{..........................} (1).$$

The equation of continuity is $\dfrac{d\rho}{dt} + \dfrac{d(\rho u)}{dx} = 0$, or, substituting for ρ, and retaining only the first powers of small quantities,

$$\frac{ds}{dt} + \frac{d^2\phi}{dx^2} = 0 \quad \text{..........................} (2).$$

Hence, from (1) and (2),

$$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2}, \text{ where } k\beta = a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

is the equation which determines the oscillatory motions of the air in a straight tube.

The integral of this equation is

$$\phi = F_{,}(x+at) + f_{,}(x-at);$$

hence $$u = F(x+at) + f(x-at),$$

taking F and f as the derived functions of $F_{,}$ and $f_{,}$,

$$\text{and } s = -\frac{1}{a^2}\{aF(x+at) - af(x-at)\},$$

$$as = -F(x+at) + f(x-at).$$

The initial circumstances of motion will determine these functions.

Initially, when $t = 0$, let

$$u = \psi(x) \text{ and } s = \chi(x),$$

$$\text{then } F(x) + f(x) = \psi(x),$$

$$\text{and } F(x) - f(x) = -a\chi(x).$$

$$\left.\begin{aligned}\text{Hence } 2F(x) &= \psi(x) - a\chi(x)\\ 2f(x) &= \psi(x) + a\chi(x)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

$$\therefore 2u = \psi(x+at) - a\chi(x+at) + \psi(x-at) + a\chi(x-at),$$

$$2as = -\psi(x+at) + a\chi(x+at) + \psi(x-at) + a\chi(x-at);$$

and, if the functions $\psi(x)$ and $\chi(x)$ are given for all values of x from $-\infty$ to $+\infty$, the values of u and s are determined for all values of x and t.

In order to express the actual motion of a particular element of the fluid we must replace u by $\frac{dx}{dt}$, and obtain x in terms of t.

188. The equations of the preceding article may be also obtained without reference to the general equation of motion.

Let A be the area of a transverse section of the tube, x and

$x + \delta x$ the distances from the origin of two particles near one another when at rest,

$$x + \xi,\ x + \delta x + \xi + \frac{d\xi}{dx}\delta x$$

the distances of the same particles when in motion at the time t.

Hence if ρ_0 be the density of the fluid in the space δx, and ρ of the same fluid when occupying the space $\delta x + \frac{d\xi}{dx}\delta x$,

$$\frac{\rho}{\rho_0} = 1 + s = \frac{\delta x}{\left(1 + \frac{d\xi}{dx}\right)\delta x} = 1 - \frac{d\xi}{dx} \text{ approximately,}$$

$$\text{or } s = -\frac{d\xi}{dx}.$$

If p be the pressure at the distance $x + \xi$, that is, the pressure at the time t about the particle whose distance when at rest is x,

$p + \frac{dp}{dx}\delta x$ is the pressure at the distance $x + \delta x + \xi + \frac{d\xi}{dx}\delta x$.

The moving force on the mass $A\rho_0\delta x = -A\frac{dp}{dx}\delta x$, and

$$\therefore\ A\rho_0\delta x\frac{d^2\xi}{dt^2} = -A\frac{dp}{dx}\delta x.$$

But $$p = k\rho_0(1 + \beta s) = k\rho_0\left(1 - \beta\frac{d\xi}{dx}\right),$$

$$\therefore\ \frac{dp}{dx} = -k\rho_0\beta\frac{d^2\xi}{dx^2},$$

$$\text{and } \frac{d^2\xi}{dt^2} = k\beta\frac{d^2\xi}{dx^2} = a^2\frac{d^2\xi}{dx^2}.$$

The integral of this equation is of the form

$$\xi = \phi(x + at) + \psi(x - at),$$

$$\text{and } \therefore\ u = \frac{d\xi}{dt} = a\phi'(x + at) - a\psi'(x - at),$$

$$s = -\frac{d\xi}{dx} = -\phi'(x + at) - \psi'(x - at),$$

which are the same as the results of the preceding article if we write $F(x+at)$ for $a\phi'(x+at)$, and $f(x-at)$ for $-a\psi'(x-at)$.

189. The equation for aerial vibrations,

$$\frac{d^2\xi}{dt^2} = k\beta\frac{d^2\xi}{dx^2},$$

is an approximate equation; an exact equation can be obtained as follows.

By the principles of thermodynamics, or by Art. 93, if p be the pressure and ρ the density of a quantity of air, and if the loss of heat by radiation or conduction be insensible,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\beta.$$

Now $$\rho = \rho_0(1+s) = \frac{\rho_0}{1+\dfrac{d\xi}{dx}},$$

and the equation of motion is

$$\frac{d^2\xi}{dt^2} = -\frac{1}{\rho_0}\frac{dp}{dx};$$

$$\therefore \frac{d^2\xi}{dt^2} = -\frac{p_0}{\rho_0}\frac{d}{dx}\frac{1}{\left(1+\dfrac{d\xi}{dx}\right)^\beta}$$

$$= \beta k \frac{\dfrac{d^2\xi}{dx^2}}{\left(1+\dfrac{d\xi}{dx}\right)^{\beta+1}},$$

or, if $$x+\xi = y \text{ and } \beta k = a^2,$$

$$\left(\frac{dy}{dx}\right)^{\beta+1}\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}.$$

To find a solution of this equation,

let $$\frac{dy}{dt} = f\left(\frac{dy}{dx}\right).$$

Then $$\frac{d^2y}{dt^2} = f'\left(\frac{dy}{dx}\right)\frac{d^2y}{dxdt} = \left\{f'\left(\frac{dy}{dx}\right)\right\}^2\frac{d^2y}{dx^2};$$

$$\therefore f'\left(\frac{dy}{dx}\right) = \pm a\left(\frac{dy}{dx}\right)^{-\frac{\beta+1}{2}},$$

and $$\frac{dy}{dt} = f\left(\frac{dy}{dx}\right) = C \pm \frac{2a}{\beta-1}\left(\frac{dy}{dx}\right)^{-\frac{\beta-1}{2}}.$$

Integrating by Charpit's method, we obtain the general primitive as the result of the elimination of α between the equations,

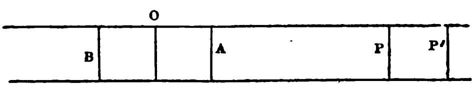
$$\left.\begin{aligned} y &= \alpha x + \left(C \pm \frac{2a}{\beta-1}\alpha^{-\frac{\beta-1}{2}}\right)t + \phi(\alpha), \\ 0 &= \alpha x \mp a\alpha - x^{\frac{\beta-1}{2}} t + \alpha\phi'(\alpha). \end{aligned}\right\}$$

Mr Earnshaw discusses these equations in a paper on the Theory of Sound, *Phil. Trans. R. S.* Vol. 150.

190. *To find the velocity with which a disturbance is propagated along the tube.*

Let the initial disturbance extend through a space AB, (2λ), from $x = -\lambda$ to $x = +\lambda$; then $\psi(x)$ and $\chi(x)$ are each zero for all values of x, except those comprised between $x = \pm\lambda$, and, from the equations (4), it appears that $F(x)$ and $f(x)$ are subject to the same law.

First, consider the motion of the fluid at a point P, such that x, i.e. OP, is greater than λ.



In this case $x + at > \lambda$, and therefore, $F(x)$ being zero except when $x > -\lambda$ and $< +\lambda$,

$$F(x + at) = 0,$$

hence $\quad u = f(x - at), \quad as = f(x - at), \quad$ and $u = as$.

Also $f(x-at)$ is zero, and therefore u and s are zero, except for values of t which make

$$x - at < \lambda \text{ and } > -\lambda;$$

so that if τ, τ', be the times at which the motion of P begins and ends,

$$x - a\tau = \lambda, \quad x - a\tau' = -\lambda, \quad \text{and } \therefore \tau' - \tau = \frac{2\lambda}{a}.$$

Hence P is set in motion at the time τ, vibrates during the time $\frac{2\lambda}{a}$, and is afterwards at rest.

Again, if P' be another point of the tube, the fluid at P' will be set in motion at a time τ_1 such that

$$OP' - a\tau_1 = \lambda;$$

and, since $OP - a\tau = \lambda$, it follows that $\frac{PP'}{\tau_1 - \tau} = a$; and therefore a is the rate at which the disturbance travels along the tube in the positive direction. When P has come to rest, the motion which P had at first will have been transmitted to a point at a distance 2λ from P, since $\frac{2\lambda}{a}$ is the time during which P is in motion.

The disturbance therefore travels along the tube in the form of a *wave* of constant length 2λ and with a constant velocity.

Secondly, consider a point on the negative side and such that $x < -\lambda$.

Hence $\qquad x - at < -\lambda,$

and therefore $\qquad f(x - at) = 0,$

$$u = F(x + at), \quad as = -F(x + at), \quad \text{and} \quad u = -as.$$

These expressions will be zero unless

$$x + at > -\lambda \text{ and } < \lambda;$$

and therefore if x, x', be the distances from the origin of two points at which motion commences at the times τ, τ', respectively,

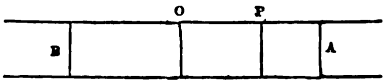
$$x + a\tau = -\lambda, \quad x' + a\tau' = -\lambda,$$
$$\text{and } x' - x = -a(\tau' - \tau),$$

from which it results, as before, that a is the velocity of propagation.

Similarly, the time of motion of any one element is $\dfrac{2\lambda}{a}$, and a wave is therefore propagated in the negative direction.

Lastly, consider the motion of a point between the limits AB of the initial disturbance.

The velocity and condensation are both given by the sum of two functions, representing respectively the disturbances due to the two waves which we have shewn to be travelling in opposite



directions. By the principle of the superposition of small motions*, the motion is therefore the same as would be caused by the coexistence of two waves, travelling across the point P in opposite directions.

Now $\qquad OP + at < \lambda$, until $t = \dfrac{\lambda - OP}{a} = \dfrac{PA}{a}$,

after which $\qquad F(OP + at) = 0$;

and $\qquad OP - at > -\lambda$, until $t = +\dfrac{\lambda + OP}{a} = \dfrac{PB}{a}$,

after which $\qquad f(OP - at) = 0$;

this is, the motion of P is represented by the coexistence of two

* The principle of the superposition of small motions asserts that if a number of small disturbing causes act on a material particle, the resulting effect is *sensibly* the sum of the effects due to each cause acting singly.

Thus, if a quantity u would be changed by one disturbing cause acting alone into $u + \alpha u$, where α is very small; and by another into $u + \beta u$, the whole change, when the two act together, will be $\alpha u + \beta u$; for if the second cause act immediately after the first, the resulting additional change would be $\beta(u + \alpha u)$ or $\beta u + \alpha\beta u$, where $\alpha\beta u$ being a small quantity of the 2nd order may be neglected in comparison with $\alpha u + \beta u$.

vibrations until the negative wave has travelled over a space AP and it is then disturbed only by the positive wave travelling over BP.

191. It may be observed that of these two waves it is possible that only one may exist; if, for instance

$$\psi(x) - a\chi(x) = 0$$

for all values of x between $\pm\lambda$, then $F(x) = 0$ for *all* values of x, and only one wave is propagated.

The function $f(x - at)$ determines the wave which is propagated in the positive direction, and, for the continuance of this motion, the relation $u = as$ is necessary. If this relation be destroyed, the wave so disturbed will give rise to two waves, one travelling in the positive and the other in the negative direction.

Definition of a Wave.

192. The term wave is applied to any state of motion transmitted through a substance the elements of which are slightly disturbed. The length of a wave is the distance between two consecutive surfaces of equal displacement, that is, two surfaces, the particles in which are, at the instant considered, in the same state of motion. In the case of the last article, a solitary wave, of the length 2λ, is propagated in the positive direction, and a similar wave in the negative direction, so that every portion of fluid in the tube has a phase of motion, and is afterwards at rest. Instances of the solitary wave may be seen in the effect of a gust of wind on a corn-field, or in the expanding circles produced by dropping a stone into still water.

If the original disturbance be repeated at the instant when the positive and negative waves, after traversing each other, have just cleared the space AB, and the process be continued, so that a series of waves follow closely on each other, then a particle of fluid, once set in motion, will vibrate, isochronously, and if a number of points be taken at successive distances 2λ, the points of division will be the positions of all the particles of fluid which, at any one instant, are in the same state of vibration.

The nature of the original disturbance will determine the character of the wave, that is, the form, extent, and rapidity of the vibrations of which it consists.

Suppose, for instance, that the air in a tube is set in motion by the oscillations of a disc, the plane of which is perpendicular to the axis of the tube. When the disc moves in the direction OA, fig. Art. (190), the air on the side A is condensed, and on the other side is rarefied: if the motion of the disc then cease, a condensing wave will be propagated in the direction OA, and a rarefying wave in the direction OB, provided the separation of the two portions of air by the disc be complete; but when this separation does not exist or when vibrations can be transmitted through the disc, a complete wave, half condensing and half rarefying, will be propagated in each direction*.

If the disc make a complete oscillation, starting from rest and returning to its original position, complete waves will be propagated in each direction.

It should be observed that the range of vibration of the disc is not necessarily comparable with the length of the wave produced; the space through which the disc oscillates may be very small compared with the space AB. In fact, whatever be the extent of the disc oscillations, the wave AB depends only on the time of the oscillations, and on the velocity of propagation.

193. *The vibrations of the air in a tube closed at one end.*

Let P be the closed end. Then if a disturbance be excited over a space AB, (2λ), it will in general cause two waves travelling in opposite directions, and one will impinge on the fixed end P.

The analytical condition is that for all values of t the value of u is zero at the point P, and we have to determine the modification introduced by this condition into our previous results.

* The term wave is sometimes applied to either of these portions; each being the distance between points of zero velocity.

Let $OP = c$; then

$$F(c + at) + f(c - at) = 0,$$

for all values of t;

$$\therefore F(x + at) = F\left\{c + a\left(t - \frac{c-x}{a}\right)\right\}$$

$$= -f\left\{c - a\left(t - \frac{c-x}{a}\right)\right\}$$

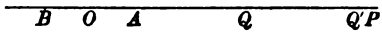
$$= -f(2c - x - at),$$

and $$u = f(x - at) - f(2c - x - at),$$

observing that $f(z)$ is zero except for values of z between $\pm\lambda$.

Now consider the motion at Q, a section of the tube between O and P.

Let $x = OQ$, and first suppose $x < c - \lambda$. When $t = \frac{x-\lambda}{a}$, Q begins to move, and from $t = \frac{x+\lambda}{a}$, i.e. after the wave has passed over Q,



$$f(x - at) = 0,$$

and $f(2c - x - at) = 0$ until $2c - x - at = \lambda$,

$$\text{or } t = \frac{2c - x - \lambda}{a} = \frac{c - \lambda + c - x}{a}$$

$$= \frac{AP + PQ}{a}.$$

$f(2c - x - at)$ is then finite, but again vanishes when

$$2c - x - at = -\lambda,$$

$$\text{or } t = \frac{2c - x + \lambda}{a} = \frac{AP + PQ + 2\lambda}{a};$$

and for all greater values of t is evanescent.

The motion of Q is therefore the same as if, when the front of the wave arrives at P, another wave, (2λ), were to start im-

mediately, and travel in the opposite direction, following the negative wave at a distance $2AP$.

In other words, the wave impinging on P is there reflected, and its motion exactly reversed.

If the distance of a point Q' from P be $< \lambda$, then for a certain time the motion of Q' will be the result of the superposition of two motions, namely, those due respectively to the incident and reflected waves; it will be caused by the reflected wave alone when $t > \dfrac{BQ'}{a}$, and will cease altogether when $t > \dfrac{BP + PQ'}{a}$.

194. *The vibrations of the air in a tube open at one end.*

Let P be the open end at a distance c from O. The air in the tube at P being in immediate communication with the atmosphere, it may be assumed that its condensation is zero. This assumption is usually made for purposes of calculation, but it appears from experiment that the point of zero condensation is at a little distance beyond the open end*.

We have then $s = 0$ when $x = c$;

$$\text{or } F(c + at) - f(c - at) = 0 \text{ for all values of } t;$$

$$\therefore F(x + at) = F\left\{c + a\left(t - \frac{c - x}{a}\right)\right\} = f\left\{c - a\left(t - \frac{c - x}{a}\right)\right\}$$

$$= f(2c - x - at),$$

$$\text{and } u = f(x - at) + f(2c - x - at).$$

By exactly the same reasoning as in Art. (190) it may be shewn that the wave on arriving at P is reflected and travels in the contrary direction with the same velocity.

195. *The vibrations in a tube of finite length.*

By the preceding investigations it appears that, if a disturbance be caused in a tube of finite length, the two waves, which

* Mr Hopkins, *On Aerial Vibrations. Camb. Phil. Trans.* 1838.

start at first in opposite directions, will be both reflected on arriving at the respective ends of the tube, and their motions reversed, whether the ends of the tube are open or closed.

$P \quad B \quad A \qquad\qquad A' \quad B' \quad Q$

If PQ be the tube and AB the portion initially disturbed, the two waves will after reflection be superimposed at $A'B'$, which is at the same distance from Q that AB is from P; and the motion as thus described will recur continually.

The time of a complete *oscillation* of the two waves will be

$$\frac{AP+PQ+QA}{a}, \text{ or } \frac{2PQ}{a}.$$

If the initial disturbance extend over the whole of the tube, the motion of any portion of air in it will always be the result of the superposition of the two motions arising from the two waves propagated in opposite directions.

196. *To find the general equation for the vibrations of an elastic fluid.*

Assuming the motion to be irrotational,

$$\frac{dp}{\rho} = -d \, . \frac{d\phi}{dt} - \frac{1}{2} dq^2, \text{ Art. (140)},$$

$$\text{and if } \rho = \rho_0 (1+s), \quad p = k\rho_0 (1+\beta s),$$

$$\text{and } \therefore \kappa\beta \log (1+s) = -\frac{d\phi}{dt} - \frac{1}{2} q^2.$$

Neglecting the squares of s and of the velocity, we obtain

$$\kappa\beta s = -\frac{d\phi}{dt}.$$

Also, substituting for ρ in the equation of continuity,

$$\frac{ds}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

$$\text{and therefore, } \frac{d^2\phi}{dt^2} = a^2 \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right),$$

$$\text{where } a^2 = \kappa\beta.$$

197. *A disturbance is excited in a homogeneous atmosphere so as to proceed symmetrically from a centre; it is required to determine the motion.*

In other words, the problem is, to determine the laws of the propagation of spherical atmospheric waves.

Taking the centre of the disturbance as the origin, the velocity (V) and condensation at any point will be functions of the distance (r) from the origin.

The equation of the previous article becomes

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right),$$

$$\text{or, } \frac{d^2(r\phi)}{dt^2} = a^2\frac{d^2(r\phi)}{dr^2};$$

r and t being here independent variables.

Hence, $r\phi = F(r+at) + f(r-at)$,

and therefore, since $V = \dfrac{d\phi}{dr}$, and $a^2 s = -\dfrac{d\phi}{dt}$,

$$\left.\begin{aligned} V &= \frac{1}{r}\{F'(r+at) + f'(r-at)\} - \frac{1}{r^2}\{F(r+at) + f(r-at)\} \\ \text{and } as &= \frac{1}{r}\{f'(r-at) - F'(r+at)\} \end{aligned}\right\} \ldots(\alpha).$$

In order to determine these functions, the initial values of V and s must be given for all values of r from 0 to ∞, and we must, besides, take account of the condition that, at the origin, $V=0$ always, a condition obviously true, if the disturbance be symmetrical with respect to the centre.

This condition will be satisfied if when r is infinitesimal,

$$F(r+at) + f(r-at) = Tr,$$

$$F'(r+at) + f'(r-at) = T,$$

T being an unknown function of t. Differentiating with

regard to t the first of these equations, making $r=0$, and putting z for at, we obtain

$$F(z)+f(-z)=0,\quad F'(z)-f'(-z)=0 \,\ldots\ldots\ldots (\beta),$$

for positive values of z only.

These equations determine $f(-z)$ and $f'(-z)$, if the values of $F(z)$ and $F'(z)$ can be found for positive values of z.

Let $\psi(r)$ and $\chi(r)$ be the initial values of V and as, these functions being given for all values of r from 0 to ∞.

Then, $\psi(r)=\dfrac{d}{dr}\dfrac{F(r)+f(r)}{r}$, $\chi(r)=\dfrac{f'(r)-F'(r)}{r}$,

and therefore,
$$\left.\begin{aligned}\frac{1}{r}F(r)+\frac{1}{r}f(r)&=\int\psi(r)dr=\psi_1(r)+b\\ f(r)-F(r)&=\int r\chi(r)dr=\chi_1(r)+c\end{aligned}\right\}\ldots\ldots(\gamma),$$

b and c being arbitrary constants.

We can now show that the constants in (γ) will disappear in the final expressions for the combinations of $F(z)$ and $f(z)$ which determine V and s.

Thus, taking account of the constants only, we obtain from (γ)

$$F(r)=\frac{1}{2}(br-c),\quad f(r)=\frac{1}{2}(br+c);$$

$$\therefore\ F(r+at)=\frac{1}{2}b(r+at)-\frac{1}{2}c,\quad F'(r+at)=\frac{1}{2}b,$$

and, if $r>at$,

$$f(r-at)=\frac{1}{2}b(r-at)+\frac{1}{2}c,\quad f'(r-at)=\frac{1}{2}b.$$

If $r<at$,

$$F(at-r)=\frac{1}{2}b(at-r)-\frac{1}{2}c,\quad F'(at-r)=\frac{1}{2}b,$$

but, from (β), if $r<at$,

$$\begin{aligned}f(r-at)&=-F(at-r)\\ &=\frac{1}{2}b(r-at)+\frac{1}{2}c,\end{aligned}$$

the same as when $r>at$.

Substituting in (α), we find $V=0$ and $s=0$; the constants may therefore be omitted, and we obtain from (γ), putting z for r,

$$\left.\begin{aligned} 2F(z) &= z\psi_1(z) - \chi_1(z), \\ 2f(z) &= z\psi_1(z) + \chi_1(z), \\ 2F'(z) &= \psi_1(z) + z\{\psi(z) - \chi(z)\}, \\ 2f'(z) &= \psi_1(z) + z\{\psi(z) + \chi(z)\}. \end{aligned}\right\} \quad (\delta).$$

By these equations, if the initial disturbance be given, the subsequent motion is determined.

Poisson, *Mécanique*, Art. 660.

198. *Determination of the velocity of propagation of a spherical wave.*

Let the initial disturbance extend from $r=0$ to $r=a$; then $\psi(r)$ and $\chi(r)$ have given values from $r=0$ to $r=a$, and are zero from $r=a$ to $r=\infty$. The integrals $\psi_1(r)$ and $\chi_1(r)$ are constant for all values of r greater than a, and, if we assume that $f(z)$ and $F(z)$ vanish when $z=\infty$, these integrals will vanish when $r=\infty$, and will therefore vanish for all values of r from a to ∞.

Hence, from the equations (δ), if $r<a$, $F(r+at)$ and $F'(r+at)$ are finite as long as

$$t<\frac{a-r}{a};$$

also $f(r-at)$ and $f'(r-at)$, are finite as long as

$$t<\frac{r}{a};$$

and, when

$$t>\frac{r}{a},$$

we find that $f(r-at)$ and $f'(r-at)$ are finite for values of t less than $\dfrac{a+r}{a}$: it appears then that the original surface of disturbance, for which $r=a$, is in motion during the time $\dfrac{2a}{a}$.

If $r>a$, $F(r+at)$ and $F'(r+at)$ are zero, and we have

$$V=\frac{1}{r}f'(r-at)-\frac{1}{r^2}f(r-at) \quad (1),$$

$$as=\frac{1}{r}f'(r-at).$$

But, from the equations (δ), $f(r-at)$ and $f'(r-at)$ are finite from

$$t=\frac{r-a}{a} \text{ to } t=\frac{r+a}{a},$$

and therefore the particles of fluid about the spherical surface of which r is the radius are in motion during the time $\frac{2a}{a}$.

Moreover, the motion of a particle at the distance r commences when $t=\frac{r-a}{a}$, and of a particle at the distance r' when $t=\frac{r'-a}{a}$, and the motion extends from the sphere r to the sphere r' in the time $\frac{r'-r}{a}$; a therefore represents the velocity with which the wave motion is propagated.

At a considerable distance from the centre of the initial disturbance, the terms involving $\frac{1}{r^2}$ may be neglected, and we obtain

$$V=as,$$

the relation before obtained in the case of vibrations in a straight tube. This result might have been anticipated, for at a considerable distance from the centre a small portion of the wave front would be approximately plane, and would therefore follow the laws of motion of a plane wave.

199. *Nature of the motion when the initial displacement is small but not symmetrical with regard to a centre.*

In the case of a spherical wave, suppose a conical surface described, of very small vertical angle, with its vertex at the centre of the sphere; we may conceive the air within this cone to be isolated, without affecting its motion in any way.

Now, whatever be the form of the initial surface of displacement, we can suppose the aerial mass divided into a number of cones having their vertices at the origin, in each of which the velocity of propagation will be the same, and of the nature of the propagation of a spherical wave. After the lapse of a finite

time, the several portions of the surface of disturbance, or wave surface, will be sensibly at the same distances from the origin, if the initial disturbance be of small extent, and therefore the wave surface will approximate, as it expands, to a spherical form.

200. *Comparison with observation of the theoretical velocity of sound.*

If D be the density af air at rest, and κD its pressure, and if h be the height of the barometer, and σ the density of mercury,

$$\kappa D = g\sigma h,$$

and the expression for the velocity $\sqrt{\kappa\beta}$, becomes

$$\sqrt{gh\beta\frac{\sigma}{D}}.$$

Now taking a foot and a second as units of length and time, $g = 32{\cdot}2$, and in dry air at the freezing temperature, the height of the barometer being 29·927 inches, the experiments of Biot give $\frac{\sigma}{D} = 10463$.

The quantity β may be determined by observations on the increase of temperature in a given mass of air produced by a given condensation. From the experiments of Clement and Desormes the value obtained is 1·3492.

Hence, the velocity of sound at the freezing temperature

$$= \left\{32{\cdot}2 \times 10463 \times 1{\cdot}3492 \times \frac{29{\cdot}927}{12}\right\}^{\frac{1}{2}},$$

which is approximately 1064 feet per second, and is less than the velocity, 1090 feet per second, given by observations*.

The discrepancy depends chiefly on the uncertainty of th value of β, as determined by direct observation; β is in fact be determined by equating the expression $\sqrt{\left(gh\frac{\sigma}{D}\beta\right)}$ to the c served velocity.

* For a list of the authorities on which this statement depends, Herschel's *Sound*, Art. 16.

201. We can obtain a general equation for the propagation of a small longitudinal displacement in an elastic medium, in which p is any given function of ρ, and deduce the velocity of sound in air as a particular case.

Thus, the equation of motion being

$$\frac{1}{\rho}\frac{dp}{dx}+\frac{Du}{dt}=0,$$

and the equation of continuity

$$\frac{1}{\rho}\frac{D\rho}{dt}+\frac{du}{dx}=0,$$

we obtain by the elimination of u,

$$\frac{D}{dt}\left(\frac{1}{\rho}\frac{D\rho}{dt}\right)=\frac{d}{dx}\left(\frac{1}{\rho}\frac{dp}{dx}\right),$$

the next equation of motion for any medium.

If we take $p=f(\rho)$, we obtain

$$\frac{D}{dt}\left(\frac{1}{\rho}\frac{D\rho}{dt}\right)=\frac{1}{\rho}f'(\rho)\frac{d^2\rho}{dx^2}+\left[\frac{1}{\rho}f''(\rho)-\frac{1}{\rho^2}\{f'(\rho)\}^2\right]\left(\frac{d\rho}{dx}\right)^2.$$

Performing the operation indicated on the left-hand side, and neglecting the squares of small quantities, we obtain

$$\frac{d^2\rho}{dt^2}=f'(\rho)\frac{d^2\rho}{dx^2}.$$

This equation represents the transmission of a vibration with the velocity $\sqrt{f'(\rho)}$ or $\sqrt{\frac{dp}{d\rho}}$.

In the particular case of air, $\frac{p}{p_0}=\left(\frac{\rho}{\rho_0}\right)^\beta$, and $\therefore \frac{dp}{d\rho}=\frac{p}{\rho}=\kappa\beta$ approximately.

202. The effect of *simultaneous disturbances from different centres* may be determined by observing that the equation,

$$\frac{d^2\phi}{dt^2}=a^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right),$$

being linear, is satisfied if we take for ϕ the sum of any number of particular solutions,

hat is, if we take for ϕ an expression of the form

$$\frac{1}{r}\{F(r+at)+f(r-at)\}$$

$$+\frac{1}{r_1}\{F_1(r_1+at)+f_1(r_1-at)\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $r, r_1, \ldots$ are the distances of a point in the fluid from the several centres of disturbance.

The condensation, $\frac{d\phi}{dt}$, is therefore the sum of the several partial condensations, and the velocities parallel to the axes at any point of the fluid, are given by the equations,

$$u=\frac{d\phi}{dx}=\frac{d\phi}{dr}\frac{dr}{dx}+\frac{d\phi}{dr_1}\frac{dr_1}{dx}+\ldots$$

$$\text{or, } u=\frac{x}{r}\frac{d\phi}{dr}+\frac{x_1}{r_1}\frac{d\phi}{dr_1}+\ldots$$

$$\text{and } v=\frac{y}{r}\frac{d\phi}{dr}+\frac{y_1}{r_1}\frac{d\phi}{dr_1}+\ldots$$

$$w=\frac{z}{r}\frac{d\phi}{dr}+\frac{z_1}{r_1}\frac{d\phi}{dr_1}+\ldots$$

where (x, y, z) $(x_1, y_1, z_1)\ldots$ are the co-ordinates of the point referred to axes originating in the several centres.

The velocity in any direction is therefore the sum of the velocities in that direction due to the partial disturbances.

203. *Reflection of a spherical wave at a fixed plane.*

Suppose that two exactly similar spherical waves proceed from two centres; that is, let the velocities and condensations in the two waves be the same, simultaneously, at the same distances from the centres, and consider the nature of the disturbance which takes place at the plane which is equidistant from the two centres.

By the preceding article, it is clear that the resultant motions of the particles at this plane will be entirely parallel to the plane; it appears moreover, by the same reasoning, that the two waves will pass through each other, and afterwards proceed, as if each alone had been originally excited, and that, if a series of pairs of waves proceed from the two centres, the disturbance at any point will result from the combination, by the superposition of small motions, of the partial disturbances.

If now a rigid plane occupy the place of the geometrical plane equidistant from the centres, one of the centres of disturbance may be removed, without any other alteration in the circumstances of the motion, and, if the rigid plane be perfectly smooth, the velocities of the aerial particles in contact with it will be entirely parallel to it, and its action upon the spherical wave will be represented by a reflected wave following exactly the same laws of propagation as the incident wave.

The Direct Refraction of Sound.

204. We have seen that a disturbance in an aerial column produces two waves travelling in opposite directions, and that in these two waves the conditions $v = as$, $v = -as$, are respectively satisfied.

If then anything occur to destroy the relation $v = \pm as$ in either wave, the effect will be the production of two new waves.

Suppose the aerial column to consist of two parts, containing different gases, and that one of the waves formed in one part impinges on the plane of separation, and thus produces a disturbance in the other part.

The states of motion of the two gases may be represented respectively by the systems of equations,

$$\left.\begin{aligned} v &= f(x-at) + F(x+at) \\ as &= f(x-at) - F(x+at) \end{aligned}\right\},$$

$$\left.\begin{aligned} v' &= \phi(x-a't) + \Phi(x+a't) \\ a's' &= \phi(x-a't) - \Phi(x+a't) \end{aligned}\right\};$$

observing that the state of motion to which f and F refer is that which exists after the commencement of the impulse of the wave.

At the plane of separation the two media must have the same motion and the same pressure;

hence, if ρ, ρ', be the densities of the two media,

$$\kappa\rho\,(1+\beta s)=\kappa'\rho'\,(1+\beta' s'),$$

β and β' being constants depending on the heat developed by compression, and therefore, since in the position of equilibrium the pressures $\kappa\rho$, $\kappa'\rho'$ are equal,

$$\beta s=\beta' s'.$$

If then $x=l$ at the plane of separation, these two conditions give

$$f(l-at)+F(l+at)=\phi(l-a't)+\Phi\,(l+a't),\;\ldots\ldots\ldots\ldots(1),$$

$$f(l-at)-F(l+at)=\mu\,\{\phi(l-a't)-\Phi(l+a't)\},\ldots\ldots\ldots(2),$$

$$\text{putting } \mu=\frac{\beta' a}{\beta a'}.$$

Let $x=\pm\alpha$ mark the range of initial disturbance; then v, v', s, and s' are all zero initially except for values of x between $\pm\alpha$.

Hence, considering the second medium,

$$\phi(x)=0,\quad\text{and } \Phi(x)=0,$$

and therefore $\Phi(x+a't)=0$ always, from $x=l$ to $x=\infty$.

The relation $v'=a's'$ is therefore established, and a single wave is propagated.

But, considering the first medium, we obtain from (1) and (2), taking account of $\Phi(l+a't)=0$,

$$as=\mu v,$$

and therefore, unless $\mu=1$, there will be a reflected wave.

CHAPTER XIV.

MUSICAL SOUNDS.

205. ANY mechanical impulse of the air, of a sufficient degree of violence and suddenness, will produce a sound, and a series of impulses, following each other with sufficient rapidity, will produce the sensation of a continued sound. If the series of impulses are variable in their character, and follow no regular law of production, the result is a *noise*, but if the impulses are of the same kind, and produced at regular intervals, the result is a *musical note.*

A sound of such a nature is defined by three characteristics; these are, the *intensity* of the sound, which depends on the extent of vibration of the aerial particles, the *pitch* of the note, which depends on the rapidity with which the successive waves impinge on the ear, and a quality by which notes of the same intensity and pitch are distinguishable from each other, and which seems to be determined by the nature of the instruments employed in the production of the sound; the word *timbre* is sometimes used to express this quality*.

The velocity of propagation being the same for waves of any length, it will be seen that the pitch of a note is determined by the length of the wave, or by the time of vibration, and is higher or lower, as the time of vibration, or the length of the wave, is less or greater.

206. *To determine the notes which can be produced from a tube closed at one end.*

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* A further distinction is sometimes made by using the word *tone*. Thus the tone of a flute is different from that of other instruments, but the qualities of the notes obtained from different flutes may be different.

We may conceive a series of similar waves produced by the rapid oscillations of a disc in the column of air, the successive oscillations being exactly similar to each other.

Suppose the disc to be at one end of the tube, the other end being closed, and that the motion of the air is steady, if such a motion be possible. By steady motion is here meant the perpetual recurrence, at any one point, of the same vibration.

Let the disc be at the origin, and take l for the length of the tube; then, since the velocity at the closed end is zero,

$$0 = F(l + at) + f(l - at) \quad \dots\dots\dots\dots\dots (1).$$

Since the vibration of the disc is regular, the velocity at the origin may be represented by a periodic function $\phi(at)$, and

$$\therefore \phi(at) = F(at) + f(-at) \quad \dots\dots\dots\dots\dots (2).$$

These two equations, if ϕ be given, determine F and f, as follows:

The equation (1) is true for all values of t, and therefore, putting $t - \frac{l}{a}$ for t,

$$F(at) = -f(2l - at),$$

$$\text{and } \phi(at) = f(-at) - f(2l - at),$$

a functional equation for the determination of f.

The function ϕ being periodic, we may hence infer that f is periodic, and that its period is the same as that of ϕ.

If λ be the length of a wave proceeding from a complete vibration of the disc, $\frac{\lambda}{a}$ is the period of ϕ, and therefore of f.

We have, generally,

$$v = F(x + at) + f(x - at);$$

$$\text{but } F(x + at) = F\left\{l + a\left(t + \frac{x - l}{a}\right)\right\}$$

$$= -f\left\{l - a\left(t + \frac{x - l}{a}\right)\right\} \text{ from (1),}$$

$$= -f(2l - at - x),$$

and therefore the points at which $v=0$ are given by the equation

$$f(x-at)=f(2l-x-at).$$

Now f remains unchanged when t is changed by any multiple of $\frac{\lambda}{a}$;

$$\therefore x-at=2l-x-at\pm m\lambda,$$

$$\text{or, } l-x=\pm m\frac{\lambda}{2}.$$

These points of zero velocity are called *nodes;* their distances from the closed end are $0, \frac{\lambda}{2}, 2\frac{\lambda}{2}, 3\frac{\lambda}{2}, \ldots$, and the distance between two consecutive nodes is half the length of a wave.

Assuming the oscillations of the disc to be exactly the same in both directions, the values of f will recur with opposite signs whenever at is changed by an odd multiple of $\frac{\lambda}{2}$.

But, if $\qquad s=0,\ F(x+at)=f(x-at),$

$$\text{or, } f(2l-at-x)=-f(x-at);$$

$$\therefore 2l-at-x=x-at\pm(2m+1)\frac{\lambda}{2},$$

$$\text{or, } l-x=\pm(2m+1)\frac{\lambda}{4}.$$

This gives a series of points of zero condensations, the distances of which from the closed end are $\frac{\lambda}{4}, 3\frac{\lambda}{4}, 5\frac{\lambda}{4}, \ldots$

These points are called *loops.*

If the length l of the tube were a multiple of $\frac{\lambda}{2}$, the origin would be a node, which is clearly impossible, and therefore the motion cannot be steady; if however the length be an odd multiple of $\frac{\lambda}{4}$, the origin will be a loop, and this is consistent with the circumstances of the motion.

Taking the origin as a loop and the closed end as a node, it is evident that the greatest value of $\frac{\lambda}{4}$ is l, and therefore the vibration of longest period which can be kept up in the tube is that for which $\lambda = 4l$. The sound thus produced is the fundamental note of the tube, or the lowest note which can be obtained from it, and the time of vibration for this note is $\frac{4l}{a}$.

A state of regular vibration is always possible when λ is such that

$$l = (2m+1)\frac{\lambda}{4},$$

and therefore the times of vibration corresponding to the notes, placed in ascending order, which can be produced from the tube, are

$$\frac{4l}{a}, \frac{4l}{3a}, \frac{4l}{5a}, \ldots$$

being in the ratios $1 : \frac{1}{3} : \frac{1}{5} : \ldots$,

Reflection at the Disc.

207. Supposing that the vibrations of the disc are maintained, we have to consider its effect on the returning wave, and for this it is sufficient to remark that the motion would be practically very small compared with the rate of propagation of the aerial vibrations it excites, and the returning wave will be reflected by it as if it were fixed. The state of vibration of any particle will therefore result from the coexistence of a number of vibrations arising from the various waves which travel backwards and forwards in the tube and which are continually reinforced by new oscillations of the disc.

It is important to observe that the continuance of the sound depends on the fact that the tube is of finite length, and not merely upon the repetition of the impulses by the disc, the effects of which are to reinforce the fading intensities of the original vibration. In fact, a vibration once produced will be

perpetually reflected at the ends of the tube, until it is destroyed by the friction of the tube, the imperfect elasticity of the closed end, or the friction of the air itself, so that if the disc were to make oscillations and then to be removed, the note would be produced but its intensity would rapidly diminish. As a matter of fact such a note would not in general be heard at all after the cessation of the disturbance.

208. PROP. *To determine the notes which can be produced from a tube open at one end.*

Suppose the vibrations excited by a disc at the other end, at which the origin is taken.

As in Art. (194), we have

$$F(l+at) = f(l-at),$$

and therefore

$$F(x+at) = F\left\{l + a\left(t + \frac{x-l}{a}\right)\right\}$$

$$= f\left\{l - a\left(t + \frac{x-l}{a}\right)\right\}$$

$$= f(2l - at - x),$$

$$\text{and } as = f(x-at) - f(2l - at - x).$$

The function being periodic, if $s = 0$, we have

$$2l - at - x = x - at \pm m\lambda,$$

and therefore $$l - x = \pm m\frac{\lambda}{2},$$

giving a series of loops at distances $m\frac{\lambda}{2}$ from the open end.

If then $l = m\frac{\lambda}{2}$, a series of notes can be obtained of which the times of vibration are

$$\frac{2l}{a},\ \frac{2l}{2a},\ \frac{2l}{3a},\ \frac{2l}{4a},\ \dots$$

It may be noticed that the time of vibration of the funda-

mental note of the open tube is half that of the fundamental note of a closed tube of the same length, and the note is therefore an octave higher.

Other notes than the harmonics just discussed can be obtained from tubes by making apertures at different points, and thus establishing communications with the external air. If, for instance, at a distance c from the end at which the disturbance is excited, an aperture be made of sufficient size, the air within the tube can only vibrate steadily when this aperture coincides with the position of a loop, and therefore $2c$ will be the longest possible wave, and $\frac{2c}{a}$ the time of vibration of the lowest possible note. The other portion of the tube will be inoperative, unless indeed its length be a multiple of $2c$, in which case it might be anticipated that the air within it would vibrate in unison with the air in the length c, and thus perhaps increase the intensity of the sound.

By properly placing apertures, the notes of the diatonic scale and their harmonics, can thus be produced from a single tube*.

The construction of a *flute* is an illustration of the preceding theory: it must be observed, however, that on account of the small size of the apertures and the difficulty referred to in Art. 194, the distances of the apertures from the ends are not exactly the same as would be given by the theory.

209. *Case of a tube closed at both ends.*

The effect of an aerial disturbance in a tube closed at both ends is given by the results of Arts. 193 and 195.

To find the notes producible from the tube, we must consider the conditions necessary for steady motion; and it is clear that for this purpose the two ends must be nodes, and that, if

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* The ratios of the times of vibration corresponding to a set of notes in the diatonic scale are

$$1, \frac{8}{9}, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{8}{15}, \frac{1}{2},$$

ending at the octave of the first note.

l be the whole length of the tube, and λ the length of the wave, $\frac{l}{2}$ must be an odd multiple of $\frac{\lambda}{4}$. The times of the vibrations of the notes which can be produced are therefore

$$\frac{2l}{a}, \frac{2l}{3a}, \frac{2l}{5a}, \ldots$$

the same as for a tube open at one end.

We may suppose the disturbing cause to take effect at an opening, or *embouchure*, at the middle of the tube.

Case of a tube open at both ends.

The condition necessary for steady motion is that the two ends should be loops, and this case is therefore at once reduced to that of a tube open at one end and having the disturbance excited at the other. If, however, the embouchure be in the middle of the tube the lowest note which can be obtained will be an octave higher than if the disturbance were excited at the end.

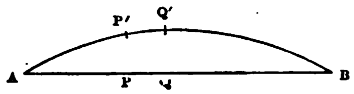
In fact, each of the two preceding cases, if the embouchure be in the middle of the tube, is equivalent to the combination of tubes, each of half the length of the tube considered; and it is easy to see that the two portions may vibrate in unison, and that, if the disturbance be excited at their plane of junction, they will do so.

The preceding investigations are applicable to the cases of tubes having a curved axis, provided the sectional area be not very large.

Organ pipes, for instance, may be bent or crooked in form, and it is found that the pitch of the note depends on the length of the axis of the tube, and is not affected by the form of the axis.

The Vibrations of Strings.

210. A piece of string or wire, tightly stretched between two fixed points, can be made to vibrate, and if the vibrations be sufficiently rapid, a musical note will be produced.



Let APB be a chord, stretched between the two points A, B, and represent by τ its tension in the position of rest.

For transversal vibrations.

Let $AP'Q'B$ be the position of the vibrating cord at the time t, $P'Q'$ being the position of the element PQ, and taking A for the origin, and AB for the axis of x, let y be the displacement of P'.

Taking θ as the inclination to the axis of x, which is very small,

$$\sin\theta = \tan\theta = \frac{dy}{dx};$$

and, if m be the mass per unit of length,

$$m\delta x\frac{d^2y}{dt^2} = \frac{d}{dx}\left(\tau\frac{dy}{dx}\right)\delta x,$$

τ remaining sensibly unchanged;

$$\therefore \frac{d^2y}{dt^2} = \frac{\tau}{m}\frac{d^2y}{dx^2},$$

representing a motion propagated with the velocity $\sqrt{\dfrac{\tau}{m}}$.

For *longitudinal vibrations*, let T be the tension at one end of the element dx when in motion;

Then, if ξ be the displacement, which may be finite,

$$m\delta x\frac{d^2\xi}{dt^2} = \frac{dT}{dx}\delta x,$$

and, by Hooke's Law, $T = E\dfrac{d\xi}{dx}$, E being the elasticity,

$$\therefore \frac{d^2\xi}{dt^2} = \frac{E}{m}\frac{d^2\xi}{dx^2}.$$

The velocity with which these vibrations move along the strings is therefore $\sqrt{\dfrac{E}{m}}$.

It may be observed that Hooke's Law is a particular case

of the general law given by Professor Maxwell, viz. that the expression $\rho\,(P-p)$ must be constant, in order that, for a finite displacement, the wave may be propagated without change of type. (*Theory of Heat*, Chapter xv.)

211. *Reflection.* It may be shewn, exactly as in the case of aerial vibrations, that any disturbance of the cord will produce two waves, travelling in opposite directions, and continually reflected at the fixed end of the chord.

Nodes and ventral segments. Putting a^2 for $\frac{T}{m}$, the transversal vibrations are given by the equation

$$y = F(x+at) + f(x-at),$$

and since, at the points A and B, $y=0$, and $\frac{dy}{dt}=0$, we have, for all values of t,

$$0 = F(at) + f(-at),$$

$$0 = F(l+at) + f(l-at);$$

$$\therefore F(x+at) = F\left\{l + a\left(t - \frac{l-x}{a}\right)\right\}$$

$$= -f\left\{l - a\left(t - \frac{l-x}{a}\right)\right\}$$

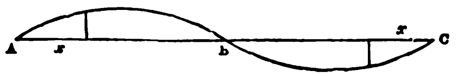
$$= -f(2l - x - at),$$

and $$y = f(x-at) - f(2l-x-at),$$

or, if $x' = 2l - x$,

$$-y = f(x'-at) - f(2l - x' - at).$$

It is inferred from this equation that if the string were continued beyond B, the displacement of a portion l would be always the same as that of AB, but in the opposite direction, and the point B would remain at rest.



The curve may evidently be continued above and below the line, and it follows therefore that, if the cord be divided into

any number of parts of equal length, regular recurrence of the same vibrations may exist in each part, and the points of division remain at rest.

These points are nodes, and the portions between them are called ventral segments.

Harmonics. The time of a complete oscillation of the whole string is $\frac{2l}{a}$, but if it be divided into ventral segments of the length $\frac{\lambda}{2}$, all the portions will oscillate simultaneously in the time $\frac{\lambda}{a}$, and the note produced will depend on λ.

The harmonics of the string are therefore given by the equation

$$n\frac{\lambda}{2} = l,$$

and the times of vibrations of the notes are

$$\frac{2l}{a}, \ \frac{2l}{2a}, \ \frac{2l}{3a}, \ \ldots\ldots\ldots$$

Coexistence of harmonics. The functions F and f being arbitrary, the equation for y may be written

$$y = F_1(x + at) + F_2(x + at) + \ldots + f_1(x - at) + f_2(x - at) + \ldots$$

or $$y = y_1 + y_2 + \ldots\ldots$$

if y_1, y_2,... be vibrations represented by the functions F_1, f_1, F_2, f_2, ... and therefore two or more vibrations of different kinds may coexist.

In practical confirmation of this result it is well known, that, besides the fundamental note of a stretched cord or wire, several of its harmonics may be heard at the same time, or indeed any number of the harmonics if the vibrations have sufficient intensity.

212. *Particular solution of the equation for transversal vibrations.*

The equation, $\dfrac{d^2y}{dt^2}=a^2\dfrac{d^2y}{dx^2}$, is satisfied if, n being an integer,

$$y=\left(A\cos\frac{n\pi at}{l}+B\sin\frac{n\pi at}{l}\right)\left(A'\cos\frac{n\pi x}{l}+B'\sin\frac{n\pi x}{l}\right),$$

which is a particular form of the general solution.

Introducing the condition that $\dfrac{dy}{dt}=0$, when $x=0$ and when $x=l$, we find that $A'=0$. If the string have initially no motion, $B=0$, and the form of the string at any time is given by the equation

$$y=\beta\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l},$$

implying that the initial form of displacement is

$$y=\beta\sin\frac{n\pi x}{l}.$$

The equation of motion is also satisfied if y be the sum of a series of terms of the same form, that is, if

$$y=\Sigma\beta\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l},$$

the values of β being assigned arbitrarily for all positive integral values of n.

In this case, if $y=f(x)$ be the initial form of displacement, we shall have

$$\Sigma\beta\sin\frac{n\pi x}{l}=f(x),$$

the value of $f(x)$ being given from $x=0$ to $x=l$.

By a known theorem, (see Todhunter's *Integral Calculus*, Chapter XIII.),

$$f(x)=\frac{2}{l}\Sigma_1^\infty\sin\frac{n\pi x}{l}\int_0^l\sin\frac{n\pi v}{l}f(v)\,dv,$$

and the values of β are determined by comparing the two series.

Thus, if β_r be the coefficient of $\sin\dfrac{r\pi x}{l}$,

$$\beta_r = \frac{2}{l}\int_0^l \sin\frac{r\pi v}{l} f(v)\, dv.$$

Suppose for an example that the string is set vibrating by pulling its middle point through a small space $\mu\frac{l}{2}$.

In this case we have, initially,

$$y = \mu x \text{ from } x = 0 \text{ to } x = \frac{l}{2},$$

$$\text{and } y = \mu(l - x) \text{ from } x = \frac{l}{2} \text{ to } x = l;$$

$$\therefore \int_0^l \sin\frac{n\pi v}{l} f(v)\, dv = \int_0^{\frac{l}{2}} \mu v \sin\frac{n\pi v}{l} dv + \int_{\frac{l}{2}}^l \mu(l - v) \sin\frac{n\pi v}{l} dv;$$

and performing the integration, we obtain

$$\beta_n = \mu\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Hence at the time t

$$\frac{\pi^2}{4l\mu} y = \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l}\cos\frac{5\pi a t}{l} - \&c.,$$

the successive terms of the expression belonging to the successive harmonics producible from the string.

In general, if initially $y = \phi(x)$, $\frac{dy}{dt} = \psi(x)$, it can be shewn that

$$y = \Sigma_{n=1}^{n=\infty} \left\{A_n \sin\left(n\pi\frac{at}{l}\right) + B_n \cos\left(n\pi\frac{at}{l}\right)\right\} \sin\left(n\pi\frac{x}{l}\right);$$

where $$A_n = \frac{2}{n\pi a}\int_0^l \psi(v) \sin\left(n\pi\frac{v}{l}\right) dv;$$

and $$B_n = \frac{2}{l}\int_0^l \phi(v) \sin\left(n\pi\frac{v}{l}\right) dv.$$

In a similar manner particular solutions can be found for the cases of aerial vibrations in closed or open tubes.

213. *The vibrations of a stretched plane membrane.*

Take the case of a plane membrane stretched tightly in two directions at right angles, and let T, T' be the tensions in these directions.

The axes of x and y being parallel to these directions, let x, y, z be the co-ordinates of a point of the membrane when slightly disturbed. Then x and y differ from their values in the position of rest by small quantities of the second order, and the equation of motion of a small element $\delta x . \delta y$ is

$$e\rho \delta x \delta y \frac{d^2 z}{dt^2} = \frac{d}{dx} (T \delta y \sin \theta) \delta x + \frac{d}{dy} (T' \delta x \sin \theta') \delta y,$$

e being the thickness and ρ the density of the membrane, and θ, θ' the inclinations to the plane xy of the tangents at the point (x, y, z) which are in the planes $Y = y$, $X = x$, respectively.

But $\sin \theta = \tan \theta = \frac{dz}{dx}$, and $\sin \theta' = \tan \theta' = \frac{dz}{dy}$;

$\therefore$ the equation of motion becomes

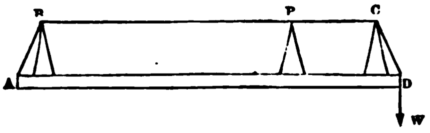
$$e\rho \frac{d^2 z}{dt^2} = T \frac{d^2 z}{dx^2} + T' \frac{d^2 z}{dy^2}.$$

Taking $T = T'$, we fall upon the case of the vibrations of the membrane of a *drum*, and, measuring r from the centre of the drum-head, the equation may be transformed into

$$e\rho \frac{d^2 z}{dt^2} = T \left(\frac{d^2 z}{dr^2} + \frac{1}{r} \frac{dz}{dr} \right).$$

The Monochord.

214. The results of theory may be tested by this instrument, which in its simplest form consists of a piece of wire or catgut fastened at one end, and stretched over two fixed edges B and C, fixed to a sounding-board, by a weight at the other



end. Between the points B, C, is a moveable bridge, by means of which any point of the string can be reduced to rest, and therefore by varying the weight and the position of the bridge any note can be produced.

This apparatus may be employed to determine the rates of vibration of musical notes. Thus, if the bridge be moved until the fundamental note of BP is the same as any particular note, that is, in unison with it, and if $BP = l$, the time of vibration

$$= 2\sqrt{\left(\frac{lw}{g\tau}\right)},$$

where w is the weight of l, and τ the tension. The length l can be obtained from a graduated scale on AD, and τ is the weight suspended to the string.

For the exemplification of the theory of harmonics it is convenient to have two wires of the same substance, and of the same length, fastened to the sounding-board, and for this purpose it is not necessary to produce the tension by means of a weight. The wires may be tightened by screws, and equality of tension can be secured by sounding their fundamental notes. The moveable bridge may be so constructed as to be in contact with one wire and not with the other.

Longitudinal Vibrations of Rods.

215. Suppose that vibrations are excited in a straight rod which is slightly compressible and slightly extensible in the direction of its axis, and that the motion of every particle in the same normal section is the same and in direction of the axis.

Let AB be the axis of the rod, the end A being fixed. Taking A for the origin, let x be the distance of a particular section P from A when undisturbed, and $x + \xi$ its distance AP' at the time t, $PQ = \delta x$, the length of an element, and T the tension, or resistance to compression, at P'.

The equation of motion for small displacements is therefore

$$m\delta x \frac{d^2\xi}{dt^2} = \frac{dT}{dx}\delta x,$$

m being the mass per unit of length.

Assuming Hooke's Law both for extension and compression,

$$T = E\frac{d\xi}{dx},$$

and therefore

$$\frac{d^2\xi}{dt^2} = \frac{E}{m}\frac{d^2\xi}{dx^2}.$$

If $a\alpha$ be the extension produced in a length a by a force W,

$$a\alpha = a\frac{W}{E}, \quad \text{and } \frac{E}{m} = \frac{W}{m\alpha}.$$

If both ends of the rod are fixed, we have $u = 0$, and $\frac{d\xi}{dt} = 0$, when $x = 0$, or l, and, comparing tension with condensation, there is an exact analogy between this case and that of a column of air in a tube closed at both ends. If one end is free,

$$T = 0, \text{ or } \frac{d\xi}{dx} = 0,$$

when $x = l$, and the analogous case is that of air in the tube open at one end.

The times of vibration of the fundamental notes in these two cases are therefore $\frac{2l}{c}$ and $\frac{4l}{c}$ respectively, c being $\sqrt{\frac{E}{m}}$.

The Propagation of Sound through Liquids.

216. Liquids, it is well known, are not absolutely incompressible, but, as very great force is required to produce a sensible compression, it is sufficient in all ordinary cases to neglect the change produced in the volume of a compressed liquid.

The great elasticity of water and other liquids renders however such media more capable of transmitting small vibratory motions than the air, and the velocity of propagation is in fact more than four times that of sound in air.

Suppose vibrations propagated along a column of water confined in a straight tube; substituting the compressibility of water for that of the rod, the case is the same as that of

Art. (215), and therefore if αa be the compression of a length a of the column produced by a force W, the velocity of propagation is given by the expression $\sqrt{\frac{W}{m\alpha}}$.

217. It has been found that an additional pressure of one atmosphere produces in water, at the freezing temperature, a compression given by $\alpha = \cdot 000049589$; that is, αl is the diminution produced in the height of a column l of water by the weight of a column of mercury 29·927 inches in height.

If ρ be the density of mercury, σ that of water, and if κ be the sectional area of the column, then, taking a foot as the unit of length,

$$m = \sigma\kappa, \text{ and } W = \frac{1}{12} g\rho\kappa \, (29\cdot 927);$$

$$\text{and the velocity of sound in water} = \left\{\frac{1}{12}\frac{g\rho}{\sigma\alpha}(29\cdot 927)\right\}^{\frac{1}{2}}$$

$$= \left(\frac{32\cdot 2 \times 29\cdot 927 \times 13\cdot 595}{12 \times \cdot 00004958}\right)^{\frac{1}{2}},$$

since, at the freezing temperature, $\frac{\rho}{\sigma} = 13\cdot 595$.

Calculating by logarithmic tables the value of this expression, we obtain, as the velocity of sound in water, 4693 feet per second.

By experiments made in the Lake of Geneva in 1826, the velocity of sound was found to be 4708 feet per second, the temperature of the water being about 8° *C.*

The compressibility of the water, at the freezing point, was found to be the same as at the temperature 8° *C*, and therefore the quantity α may be considered as unaffected by a change of temperature. Moreover, since the atmospheric pressure employed is that of a *standard* atmosphere, the quantity ρ (29·927) is also unaffected by a change of temperature.

The only element then which can vary in the expression for the velocity is the density of water, and, as this density is a

maximum for a temperature of about $4^\circ C$, it may be anticipated that the densities at 0° and at $8^\circ C$ will be very nearly the same.

Such in effect is the case, the densities of water at 0°, 4°, and 8°, being in the ratios ·999873 : 1 : ·999878.

The velocity of sound should therefore be very nearly the same at $8^\circ C$ as at $0^\circ C$, and we must look for the causes of the discrepancy noticed above in the presence of extraneous substances in the water, and in the numerous errors to which observations of such a kind are necessarily liable.

It may be noticed that the heat developed by compression does not appear to affect in a sensible degree the velocity of sound in water.

218. When two strings of the same kind, very nearly in unison with each other, are set in vibration together, an intermitting sound is produced, and the alternations of intensity follow each other at regular intervals. If, for instance, the two strings belonging to any note of a pianoforte are not quite in unison, the note heard is alternately loud and faint*. Such alternations of intensity are called *beats*, and the more nearly the strings are in unison the greater is the interval between the beats.

Let τ, τ' be the times of vibration in fractions of a second of two strings, and suppose the vibrations to commence in the same phase; at first, the two vibrations will reinforce each other, but the faster string will gain on the other, until the vibrations are in opposite phases, in which case the two vibrations will partially destroy each other, and if the strings are exactly alike, and nearly in unison, there will be an instant of almost perfect silence, after which the vibrations will again gradually reinforce each other.

Let the faster gain one vibration in x seconds; then

$$x\left(\frac{1}{\tau} - \frac{1}{\tau'}\right) = 1, \text{ or } x = \frac{\tau\tau'}{\tau' - \tau},$$

* If there are three strings to a note, as is frequently the case, there will be a triple series of beats, arising from each pair of strings.

which is the period of the beats, and is evidently greater, as $\tau' - \tau$ is smaller in comparison with τ or τ'.

It is not essential to the production of beats that the two strings should be nearly in unison; beats will also be heard when two strings form very nearly a concord. Suppose for instance in the case of a perfect fifth, in which the ratio of the vibrations should be 3 : 2, that one string makes 201 vibrations while the other makes 300; then about the 100th of the first or the 150th of the second, the former will have gained half a vibration on the other, and the two will be opposed.

Beats will result which are very distinctly marked; and in a similar manner beats can be obtained from other concords.

The earliest notice of these sounds is by Sauveur, about 1700. Their theory is given in Smith's *Harmonics*, a treatise published in 1749.

Of a different nature are the *resultant sounds* which are sometimes heard when the concord of two notes is perfect. In the case of a perfect fifth every second vibration of one coincides with every third of the other, and the effect produced is that of a note exactly one octave below the lowest note of the concord.

These sounds, called Tartini's beats, are discussed in a treatise, by Tartini, dated 1754. The term *subharmonics* has been applied to them by musical writers.

In general, if in a certain fraction (τ) of a second, one note makes m vibrations and the other n, the period of vibration of the resultant subharmonic is τ, m and n being supposed prime to each other.

219. *Limits of audibility.* Any aerial disturbance of sufficient intensity will produce a sound of some kind, but for the production of a note or continuous sound, it is necessary that periodical vibrations should recur with a certain degree of rapidity. It is stated by writers on music that when the number of vibrations is less than 16 per second, the successive impulses are separately appreciated, and the sensation of a continuous

sound therefore implies that the number is greater than 16 per second.

On the other hand, if the number of vibrations be greatly increased, that is, if the pitch of the note be very much raised, the sound becomes gradually faint, and beyond a certain limit is quite lost. This limit varies for different persons, but the general range of human hearing from the lowest note of an organ to the highest appreciable sound appears to be about nine octaves.

Hence the times of vibration of the extreme notes, and therefore the lengths of the corresponding waves, are approximately in the ratio $2^9 : 1$.

If a be the velocity of sound, and l the length of an organ-pipe, its fundamental note arises from vibrations of which the period is $\frac{2l}{a}$, and, equating this to $\frac{1}{16}$,

$$\text{we obtain } l = \frac{a}{32} = \frac{1090}{32} = 34 \text{ feet nearly,}$$

and the wave length is therefore about 68 feet.

Hence the shortest wave length is about $\frac{68}{2^9}$ feet or 1.6 inches.

From a series of experiments performed by Savart, it appears that this range may be extended, and that the limits of sensibility of the ear are frequently separated by eleven octaves.

In taking 1090 feet per second as the velocity of sound, we have supposed that the temperature is near the freezing point; if the temperature be greater the velocity is greater, and the length l may therefore be increased, or, if l be given, the time of vibration will be diminished. This is in accordance with the known fact that the pitch of an ordinary open organ-pipe is raised by an increase of temperature.

CHAPTER XV.

LIQUID WAVES.

220. If we consider the case of heavy liquid in a state of oscillation, and if we imagine the disturbances so small that the squares of the velocities may be neglected, the equation of motion, assuming it to be irrotational, is

$$\frac{p}{\rho} + \frac{d\phi}{dt} = gy,$$

measuring y vertically downwards from the free surface when the liquid is in equilibrium. This equation, combined with the equation of continuity,

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

determines the small oscillations.

221. *Oscillations of water in a straight canal of uniform depth and width.*

Measuring x in direction of the length of the canal the equation of continuity is

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \quad\text{.......................(1).}$$

Taking p as the excess of the pressure of the liquid over atmospheric pressure, we have at the surface

$$\frac{d\phi}{dt} = gy, \text{ and } \therefore \frac{d^2\phi}{dt^2} = g\frac{dy}{dt} = g\frac{d\phi}{dy} \quad\text{...........(2).}$$

If h be the depth,

$$\frac{d\phi}{dy} = 0, \text{ when } y = h, \text{ for all values of } x \quad\text{........(3).}$$

The equation (1) is satisfied by

$$\phi = (A\epsilon^{my} + B\epsilon^{-my})\,(A'\cos mx + B'\sin mx),$$

A, B, A', and B' being functions of the time.

From the condition, (3),

$$0 = A\epsilon^{mh} - B\epsilon^{-mh},$$

and we may write

$$\phi = A\,(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\cos m\,(x-\gamma).$$

The equation (2) will be satisfied by taking m and γ constant, and by assuming

$$\frac{d^2A}{dt^2} = -a^2A,$$

where $$a^2 = mg\,\frac{\epsilon^{mh} - \epsilon^{-mh}}{\epsilon^{mh} + \epsilon^{-mh}}.$$

Hence it follows that

$$A = C\cos at + D\sin at,$$

and therefore

$$\phi = (C\cos at + D\sin at)\,(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\cos m\,(x-\gamma)\ldots(4),$$

a particular solution.

Changing the origin this may be written,

$$\phi = (\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\,\{A\cos(mx - at) + B\cos(mx + at)\},$$

representing two waves travelling in opposite directions.

Taking the positive wave,

$$\phi = (\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\,A\cos(mx - at),$$

and, at any instant, ϕ is the same at the ends of a range $\frac{2\pi}{m}$,

$\therefore$ λ being the wave length,

$$\lambda = \frac{2\pi}{m},$$

and the time (τ) of an oscillation $= \frac{2\pi}{a}$.

Now, if v be the velocity of propagation,

$$\lambda = v\tau,$$

$$\therefore\ v = \frac{a}{m} = \sqrt{\frac{g}{m} \cdot \frac{\epsilon^{mh} - \epsilon^{-mh}}{\epsilon^{mh} + \epsilon^{-mh}}};$$

or, in terms of λ,

$$v = \sqrt{\frac{g\lambda}{2\pi} \cdot \frac{\epsilon^{\frac{2\pi h}{\lambda}} - \epsilon^{-\frac{2\pi h}{\lambda}}}{\epsilon^{\frac{2\pi h}{\lambda}} + \epsilon^{-\frac{2\pi h}{\lambda}}}}.$$

222. To find the motion of a particle, let ξ, η be the displacements of the particle at (x, y); then

$$\frac{d\xi}{dt} = \frac{d\phi}{dx} = -m\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right) A \sin(mx - at),$$

$$\frac{d\eta}{dt} = \frac{d\phi}{dy} = -m\left(\epsilon^{m(h-y)} - \epsilon^{-m(h-y)}\right) A \cos(mx - at).$$

Now if a, β be the co-ordinates of the mean position,

$$x = a + \xi, \quad y = \beta + \eta,$$

ξ and η being very small, and therefore the values of $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$ will be, approximately,

$$\frac{d\xi}{dt} = -mA\left(\epsilon^{m(h-\beta)} + \epsilon^{-m(h-\beta)}\right) \sin(ma - at),$$

$$\frac{d\eta}{dt} = -mA\left(\epsilon^{m(h-\beta)} - \epsilon^{-m(h-\beta)}\right) \cos(ma - at).$$

Integrating and combining the results, we obtain

$$\frac{\xi^2}{\left(\epsilon^{m(h-\beta)} + \epsilon^{-m(h-\beta)}\right)^2} + \frac{\eta^2}{\left(\epsilon^{m(h-\beta)} - \epsilon^{-m(h-\beta)}\right)^2} = \frac{m^2A^2}{a^2};$$

the particles therefore describe ellipses about their mean positions, the axes of which are in the ratio

$$\epsilon^{m(h-\beta)} + \epsilon^{-m(h-\beta)} : \epsilon^{m(h-\beta)} - \epsilon^{-m(h-\beta)}.$$

As β is taken from 0 to h, these ellipses gradually flatten into straight lines.

If h be very great, the ratio approximates to a ratio of equality, and the ellipses become circles.

223. If h be small compared with λ, as in the case of long waves, the equation for the velocity of propagation becomes

$$v = \sqrt{gh}.$$

If the depth be very great as in the case of waves in a deep sea, $v = \sqrt{\dfrac{g\lambda}{2\pi}}$.

At the surface, $\dfrac{d\phi}{dt} = gy$,

and the form of the surface is therefore given by the equation,

$$gy = Aa\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right)\sin(mx - at),$$

which, to the same approximation as before, represents a curve of sines.

224. In the preceding investigation we have assumed that the motion is irrotational, and this will be the case if the liquid be frictionless, and originally at rest. In the *Philosophical Transactions* for 1863, Professor Rankine discusses the propagation of waves in deep water when the motion is rotational, and obtains the velocity of propagation from elementary considerations*.

Rankine's theory is independent of the size of the waves, and his equations are exact, but the liquid must be supposed to have, at every point, the proper amount of molecular rotation.

Expressed analytically the co-ordinates of a particle are given by the equations,

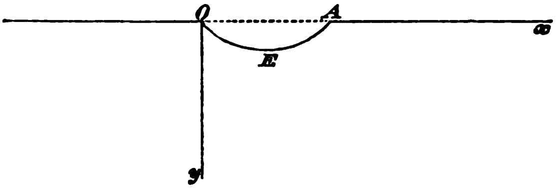
$$x = \alpha + c\epsilon^{-\frac{\beta}{c}} \sin\left(at + \frac{\alpha}{c}\right), \quad y = \beta + c\epsilon^{-\frac{\beta}{c}} \cos\left(at + \frac{\alpha}{c}\right),$$

α, β being the co-ordinates of the mean position. These equations satisfy the equation of continuity, and represent a rotational motion, the angular velocity being $a\left(\epsilon^{\frac{2\beta}{c}} - 1\right)^{-1}$.

* Rankine states in this paper that Mr Froude had also arrived at similar results by a similar process.

225. In Article 221 we obtain a particular solution of the equations of motion, and we consider the propagation of a single wave without any reference to the process of starting the motion.

Imagine that, primarily, the liquid in the canal is at rest, and that a solid cylinder of given shape is partially immersed with its axis horizontal, and perpendicular to the direction of the canal. If this cylinder be suddenly removed, without disturbing the liquid, wave motions will ensue, of an irrotational character, since the liquid initially has no motion.



Let OEA be a vertical section of the cylinder by the plane xy, and let $y = f(x)$ be the equation to the curve OEA. Then at the surface we have, initially,

$$\frac{d\phi}{dt} = gf(x), \text{ from } x = 0 \text{ to } x = l,$$

l representing OA, and

$$\frac{d\phi}{dt} = 0, \text{ for all other values of } x.$$

There being no initial motion,

$$\frac{d\phi}{dx} = 0, \text{ and } \frac{d\phi}{dy} = 0, \text{ when } t = 0,$$

$\therefore$ in the equation (4), Art. 221, $C = 0$, and the general value of ϕ is

$$\phi = \Sigma D \sin at \left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right) \cos m(x - \gamma).$$

Initially, at the surface,

$$\frac{d\phi}{dt} = \Sigma aD \left(\epsilon^{mh} + \epsilon^{-mh}\right) \cos m(x - \gamma) = gf(x).$$

By a known formula,

$$f(x) = \frac{1}{l}\int_0^l f(v)\,dv + \frac{2}{l}\Sigma_1^\infty \cos\frac{n\pi x}{l}\int_0^l f(v)\cos\frac{n\pi v}{l}\,dv,$$

and, comparing the two series, we may take

$$\gamma = 0, \text{ and } m = \frac{n\pi}{l}.$$

Hence

$$\frac{aD}{g}(\epsilon^{mh} + \epsilon^{-mh}) = \frac{2}{l}\int_0^l f(v) \cos \frac{n\pi v}{l}\, dv,$$

and therefore

$$\phi = \frac{2g}{al}\Sigma_1^\infty \frac{\epsilon^{\frac{n\pi(h-y)}{l}} + \epsilon^{-\frac{n\pi(h-y)}{l}}}{\epsilon^{\frac{n\pi h}{l}} + \epsilon^{-\frac{n\pi h}{l}}} \sin at \cos \frac{n\pi x}{l} \int_0^l f(v) \cos \frac{n\pi v}{l}\, dv,$$

where
$$a^2 = \frac{n\pi g}{l} \frac{\epsilon^{\frac{n\pi h}{l}} - \epsilon^{-\frac{n\pi h}{l}}}{\epsilon^{\frac{n\pi h}{l}} + \epsilon^{-\frac{n\pi h}{l}}}.$$

226. The theory of long waves in a canal of uniform width, and of inconsiderable depth, may be treated independently, as in Professor Stokes's Paper on Waves in the *Mathematical Journal* for 1849. If the length of the wave be large in comparison with the depth of the canal, we may neglect the vertical motions in comparison with the horizontal motions, and assume that, across any vertical section of the canal perpendicular to its length, the horizontal velocity is uniform. On this hypothesis the pressure at any depth δ below the surface will be $g\rho\delta$. Measuring x horizontally, let η be the elevation of the surface at the distance x'; then the elevation at the distance $x' + dx'$

$$= \eta + \frac{d\eta}{dx'}dx'.$$

Considering a small horizontal cylinder PP' of liquid of length dx', the difference of the pressures at its two ends P and P' will be

$$-g\rho \frac{d\eta}{dx'}dx'.$$

Now, if x be the abscissa of the vertical plane of particles through P in its position of equilibrium, and if ξ be the horizontal displacement of this plane of particles,

$$x' = x + \xi,$$

and the horizontal acceleration is $\dfrac{d^2\xi}{dt}$.

If κ be the cross section of the cylinder PP', the mass of the cylinder is $\rho\kappa dx'$, and the equation of motion is

$$\rho\kappa dx' \frac{d^2\xi}{dt^2} = -g\rho\kappa \frac{d\eta}{dx'} dx',$$

or

$$\frac{d^2\xi}{dt^2} = -g\frac{d\eta}{dx'}.$$

Neglecting the squares of small quantities, this becomes

$$\frac{d^2\xi}{dt^2} = -g\frac{d\eta}{dx}.$$

We have now to form the equation of continuity.

Let A be the area of the cross section of the canal, b the breadth at the surface.

In the position of equilibrium the volume of liquid between the planes x and $x + dx$ is Adx.

At the time t the distance between the bounding planes of this quantity of liquid is

$$dx + \frac{d\xi}{dx} dx,$$

and the area of the cross section of the liquid is approximately

$$A + b\eta;$$

$$\therefore \quad (A + b\eta)\left(1 + \frac{d\xi}{dx}\right) dx = Adx.$$

Hence the equation of continuity is

$$A\frac{d\xi}{dx} + b\eta = 0,$$

and we therefore obtain, for the equation of motion,

$$\frac{d^2\xi}{dt^2} = \frac{gA}{b}\frac{d^2\xi}{dx^2}.$$

If $\frac{gA}{b} = a^2$, the solution of this equation is

$$\xi = f(x - at) + F(x + at),$$

representing two waves travelling in opposite directions.

The case of long waves in a variable canal of small width

and depth is considered in one of Green's Papers, and Green's method may be employed in dealing with the simple case of a canal of uniform cross section.

227. It may be instructive to treat independently the case of periodic waves in a deep sea.

Measuring y vertically downwards from the surface of equilibrium, the equations are

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0,$$

and $$\frac{p}{\rho} + \frac{d\phi}{dt} = gy.$$

At the surface, $p = 0$,

$$\therefore\ gy = \frac{d\phi}{dt},$$

and $$\frac{d^2\phi}{dt^2} = g\frac{d\phi}{dy} \quad\dots\dots\dots\dots\dots\dots (\alpha).$$

If we assume

$$\phi = A\epsilon^{-\frac{2\pi y}{l}} \sin\frac{2\pi}{\lambda}(vt - x),$$

the equation of continuity is satisfied, and, when $y = \infty$, $\frac{d\phi}{dx} = 0$, and $\frac{d\phi}{dy} = 0$, as it might be expected would be the case.

At the surface the equation (α) gives

$$g\left(-\frac{2\pi}{\lambda}\right) = -\frac{4\pi^2}{\lambda^2}v^2,$$

or $$v = \sqrt{\frac{g\lambda}{2\pi}}.$$

Further, if ξ and η be the displacements, and (x, y) be regarded as the mean position,

$$\xi = -\frac{A}{v}\epsilon^{-\frac{2\pi y}{l}} \sin\frac{2\pi}{\lambda}(vt - x),$$

and $$\eta = \frac{A}{v}\epsilon^{-\frac{2\pi y}{l}} \cos\frac{2\pi}{\lambda}(vt - x);$$

$$\therefore\ \xi^2 + \eta^2 = \frac{A^2}{v^2} \epsilon^{-\frac{4\pi y}{l}},$$

that is, the particles revolve in circles, diminishing in size with the increase of depth.

It can be easily shewn, from the preceding equations, that the pressure at each particle is the same as if the liquid were still, and that the energy of a wave is half kinetic and half potential.

228. Professor Stokes proceeds to a higher degree of approximation, and arrives at the conclusion that in addition to the motion of oscillation the particles are transferred forwards, that is, in the direction of propagation, with a constant velocity depending on the depth, and decreasing rapidly as the depth increases.

As a second approximation, Professor Stokes obtains for the displacements the equations,

$$\xi = a\epsilon^{-my} \sin m\,(x - ct) + m^2 a^2 ct\epsilon^{-2my},$$
$$\eta = a\epsilon^{-my} \cos m\,(x - ct),$$

from which the statement just made follows at once.

Lord Rayleigh has shewn, by an argument of a very elementary character, that this superficial motion is an immediate consequence of the absence of molecular rotation. The fact can be observed by watching the progress of waves on the sea, for the crest of a wave will be seen advancing more rapidly than the hollow until it breaks, and then a fresh wave is formed, and the process is repeated.

229. The preceding articles deal with a simple case of the oscillations of a liquid; the following references, giving some of the sources of information on the subject, may be useful.

The article on Tides and Waves, by Sir G. B. Airy, in the *Encyclopædia Metropolitana.*

Professor Stokes, on Waves, in the *Mathematical Journal* for 1849.

Professor Stokes on Waves, in the eighth volume of the Cambridge *Philosophical Transactions.*

Rankine's Papers in the *Transactions of the Royal Society.*

Maxwell's *Heat,* Chapter XV.

Lord Rayleigh, on Waves, in the *Philosophical Magazine* for April, 1876.

MISCELLANEOUS PROBLEMS.

1. A HEMISPHERICAL bowl is filled with water; if the internal surface be divided by horizontal planes into n portions, on each of which the whole pressure is the same, and h_r be the depth of the r^{th} of these planes, prove that

$$\frac{h_r}{a} = \sqrt{\frac{r}{n}},$$

a being the radius.

2. If a triangle be immersed in a liquid in a vertical plane, and if α, β, γ be the inclinations of its sides to the horizon, and x, y, z (in order of magnitude) the depths of its vertices; prove that the depth of the centre of pressure is

$$\frac{1}{2} \cdot \frac{x^4 \sin A \sin \alpha - y^4 \sin B \sin \beta + z^4 \sin C \sin \gamma}{x^3 \sin A \sin \alpha - y^3 \sin B \sin \beta + z^3 \sin C \sin \gamma}.$$

3. A rigid spherical envelope of radius a is filled with elastic fluid of mass M which is acted on by a repulsive force $= \mu\,(\text{dist.})^2$ from a point in the surface of the envelope: shew that the total normal pressure on the envelope is

$$4\mu a^2 M \frac{\int_0^1 x \epsilon^{\frac{8\mu a^3}{3\kappa} x^3} dx}{\int_0^1 \left(\epsilon^{\frac{8\mu a^3}{3\kappa} x^3} - 1\right) dx}.$$

4. A vessel is in the form of a right cone without weight, the vertical angle being 2α; the vessel is filled with liquid and then suspended by a point in the rim: if β be the inclination of the axis of the cone to the vertical, shew that

$$\cot 2\beta = \cot 2\alpha - \frac{3}{4} \operatorname{cosec} 2\alpha.$$

5. If the depths of the angular points of a triangle below the surface of a fluid be a, b, c, shew that the depth of the centre of pressure below the centre of gravity is

$$\frac{(b-c)^2+(c-a)^2+(a-b)^2}{12(a+b+c)}.$$

6. Shew that the forces represented by

$$X=\mu(y^2+yz+z^2), \quad Y=\mu(z^2+zx+x^2), \quad Z=\mu(x^2+xy+y^2)$$

will keep a mass of liquid at rest, if the density $\propto \frac{1}{(\text{dist.})^2}$ from the plane $x+y+z=0$; and the curves of equal pressure and density will be circles.

7. Two very small spheres, of the same size but different densities, are connected by a fine string and immersed in a liquid, which rotates uniformly about a fixed axis, and is not acted upon by any forces; the density of the liquid being intermediate between the densities of the spheres, find their position of relative equilibrium.

8. Find the centre of pressure upon a portion of a vertical cylinder containing liquid, the portion being such as when unwrapped to form an isosceles triangle, the base of which when forming part of the cylinder is horizontal, and the vertex at the surface of the fluid. If this portion be divided into two equal parts by a vertical plane, find the least couple which will prevent either of the parts from turning round.

9. Two cubical vessels of height a have their bases horizontal and a common vertical face, in which an aperture is cut in the form of an equilateral triangle, whose vertex is in the base and opposite side horizontal, the length of the side being a. Fitted into this aperture is a prism of length $l\,(l<a)$, which slides freely. Equal volumes of two liquids, the specific gravities of which are in the ratio 27 : 8, are poured into the respective vessels. Determine under what conditions the prism may be in equilibrium, and prove that it never can be so unless

$$l \text{ be } > \frac{2}{3}a.$$

10. A spherical shell, whose interior radius is a, is filled with liquid of uniform density ρ, and revolves with uniform angular velocity ω about the vertical diameter of the shell; shew that, if the

total normal pressure on the upper half of the shell be to that on the lower half as $m : n$, the pressure at the highest point of the liquid is

$$\rho\left\{\frac{3m-n}{n-m}\frac{ga}{2}-\frac{\omega^2a^2}{3}\right\}.$$

11. Prove that the work done in compressing a given quantity of a perfect gas originally at the pressure P and volume V to the volume v, is $PV\log\frac{V}{v}$.

12. A hollow sphere, filled with equal quantities of two liquids which do not mix, revolves uniformly about its vertical diameter, and the liquid particles are relatively at rest. Find the angular velocity when the lighter liquid just touches the lowest point in the surface of the sphere.

13. A hollow cylinder is filled with water and made to revolve about a vertical axis attached to the centre of its upper plane face with a velocity sufficient to retain it at the same inclination to the axis. Find at what point of the surface a hole might be bored without loss of fluid.

14. A mass of liquid is contained between three co-ordinate planes, each of which attracts with a force varying as the distance, and the absolute forces of attraction μ, μ', μ'' are in harmonic progression. Half an ellipsoid is fixed with its plane face against one of the co-ordinate planes, and its surface touching the other planes, its axes being parallel to the co-ordinate axes and proportional to

$$\frac{1}{\sqrt{\mu}},\ \frac{1}{\sqrt{\mu'}},\ \frac{1}{\sqrt{\mu''}}.$$

If there be not sufficient fluid quite to cover the ellipsoid, the uncovered part will be bounded by a circle.

15. A mass of liquid is subject to the mutual gravitation of its particles, and to a repulsive force tending from a plane through its centre of gravity and varying as the perpendicular distance from that plane; shew that the conditions of equilibrium will be satisfied if the surface be a prolate spheroid of a certain ellipticity, provided the repulsive force be not too great.

16. A right cylindrical vessel on a plane base contains a certain quantity of gas, which is confined within it by a disc exactly similar

and parallel to the base; shew that the pressure on the curved surface of the cylinder is independent of the position of the disc, the temperature being constant.

17. A solid generated by the revolution of the curve $y \propto x^{\frac{n}{2}-1}$ around the axis of x, floats with a portion h of the axis immersed. If the solid be depressed through $(n^{\frac{1}{n-1}} - 1)\,h$, it will, on its return, just emerge.

18. A cylindrical diving-bell is suspended with its axis vertical at a depth such that the water rises half-way up the bell: find the least distance of the centre of gravity of the bell from the centre of its upper surface, consistent with the condition that the equilibrium may be stable with reference to an angular displacement of the axis.

19. A cylinder makes vertical oscillations in a liquid contained in another cylinder, the radius of which is n times that of the former; shew that the depth of the axis immersed when in a position of rest is $gt^2n^2 \div \pi^2 (n-1)$ where t is the time of an oscillation.

20. A vessel in the form of a paraboloid with its axis vertical, contains a quantity of liquid equal in volume to that of a segment of a paraboloid, of the same latus rectum, floating in it: if this be raised till its vertex is just in the surface, and if it then sink to a depth equal to $\frac{3}{4}$ of its axis before returning, shew that the density of the liquid : that of the paraboloid :: 48 : 7.

21. A closed cylindrical vessel one foot in height is half full of water, the other half being occupied by atmospheric air; if two small apertures be made, one at the base of the cylinder and the other five inches above it, shew that the density of the air in the vessel will decrease until it is $\left(1 - \dfrac{1}{12h}\right)$ times its original value approximately, and then increase again, h being the height of a water-barometer in feet.

22. Incompressible fluid is at rest under the action of forces

$$-\frac{\mu x}{a^2},\quad -\frac{\mu y}{b^2},\quad -\frac{\mu z}{c^2},$$

respectively parallel to the axes, and a particle, the density of which is less than that of the fluid, is placed anywhere in the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = m;$$

prove that, neglecting the resistance, the velocity of the particle when crossing the surface defined by the quantity m' varies as

$$\sqrt{m' - m}.$$

23. If the particles of a mass of liquid rotating uniformly about a fixed axis, attract one another according to such a law that the surfaces of equal pressure are similar coaxial oblate spheroids; prove that the resultant attraction of a spheroid, the particles of which attract according to the same law, is the resultant of two forces perpendicular to the equator and the axis of revolution respectively, and varying as the distance of the attracted point from them.

24. A cylindrical tube, containing air, is closed at one extremity by a fixed plate, the other extremity being open; a piston just fitting the tube slides within it, and the centres of the plate and piston are connected by an elastic string, the modulus of elasticity of which is equal to the atmospheric pressure on the piston; prove that, if l be the natural length of the string, and a its length when the air between the piston and the fixed plate is in its natural state, l being less than a, the length of the string in the position of equilibrium will be $(la)^{\frac{1}{2}}$. If the piston be slightly displaced from this position, find the time of a small oscillation.

25. An elastic spherical envelope is in equilibrium when it contains air at twice the atmospheric density, and its radius is twice the natural size; if the barometer fall $\frac{1}{n}$th of an inch, find the time of a small oscillation in the magnitude of the envelope.

26. A right cone rests in a vessel containing equal depths of two given fluids, with its vertex fastened to the bottom and its axis vertical. Find the condition for stable equilibrium.

27. A straight uniform rod consisting of matter attracting as $(\text{dist.})^{-1}$ is surrounded by fluid at rest subject to its attraction only: shew that the differential equation to the meridian sections of the surfaces of equal pressure can be put in the form

$$\frac{dy}{dx} \cdot \psi + \log \frac{r}{r'} = 0,$$

r, r' being the distances of the point xy from the ends of the rod, and ψ the angle subtended by the rod at that point.

28. A right prism on a square base has another prism, also on a square base, attached to it, so that their axes are coincident and sides parallel, and the whole floats on a fluid with their common plane in the plane of floatation. If the sides of the bases of the two prisms are in the ratio 2 : 1, find their limiting heights in order that the equilibrium may be stable.

29. A heavy cube is moveable about an axis, which passes through, and bisects, the opposite sides of one face; this axis being fixed horizontally within an empty vessel, so that the cube is suspended in the position of equilibrium, find the depth to which fluid must be poured in, so as to render the equilibrium unstable, and the greatest ratio of the densities of the cube and fluid, that this may be possible.

Supposing the cube half immersed and the equilibrium stable, find the time of a small oscillation.

30. A cylinder whose axis is vertical is floating in a fluid in which the density at any point varies as the n^{th} power of the depth; the cylinder is depressed till its upper end just coincides with the surface of the fluid, and on being let go it rises just out of the fluid; shew that, when the cylinder was floating, the depth immersed was to the height of the cylinder as 1 to $(n+2)^{\frac{1}{n+1}}$.

31. If a given quantity of homogeneous matter be formed into a paraboloid of revolution and allowed to float in water with the vertex downwards, the square of the distance of the centre of gravity from the plane of floatation will be inversely proportional to the latus rectum.

32. A semicircular cylinder rests with its axis vertical in a liquid of twice its own density; if it be moveable about the line of intersection of its vertical plane face with the surface, find the condition of stability.

33. Two equal light spheres of the same substance are attached by strings of lengths r, r' to a point in the bottom of a vessel of water—they are mutually repulsive and rest at a distance x from each other: shew that the line joining them is inclined to the horizon

at $$\sin^{-1}\frac{r^2 \sim r'^2}{x\sqrt{2(r^2+r'^2)-x^2}};$$

also if $\phi(x)$ be the repulsion

$$\phi(x) = \frac{Px}{\sqrt{2(r^2 + r'^2) - x^2}},$$

P being the fluid pressure on either sphere.

34. An embankment of triangular section ABC supports the pressure of water on the side BC: find the condition of its not being overturned about the angle A when the water reaches to B, the vertex of the triangle: and shew that, when the area of the triangle is reduced to the minimum consistent with stability for a given depth of water,

$$\tan C = \frac{\sqrt{s^2 + 2s + 9}}{3 - s},$$

$$\tan A = \frac{\sqrt{s^2 + 2s + 9}}{s - 1},$$

where s is the specific gravity of the embankment.

35. A mass (M) of fluid, in which the density at any point is the sum of a given constant quantity and a quantity bearing a given constant ratio to the pressure at that point, revolves about a fixed axis with a given constant angular velocity, and is attracted to a point in that axis by a given force which varies as the distance: find the form of the free surface; and shew that its least semi-diameter (b) is determined by the equation,

$$M = m \int_0^b e^{\frac{b^2 - x^2}{c^2}} x^2 \, dx,$$

when m and c are given constants.

36. A centre of force, repelling inversely as the square of the distance, lies below the surface of a homogeneous inelastic fluid, which is also acted on by gravity and is at rest: the intensity of the force, at a point in the surface of the fluid vertically above its centre, is equal to that of gravity: prove that the external surface of the fluid has a horizontal asymptotic plane, and that the centre of force is environed by an internal cavity, the summit of which is at the external surface of the fluid.

Find the volume of the cavity in terms of its length.

37. A candle of S.G. ρ floats vertically in still water of S.G. σ. It is lighted and the flame is observed to descend towards the water with uniform velocity u, and the velocity with which the candle burns is v: prove that $v = \dfrac{\sigma u}{\sigma - \rho}$.

Prove also, that if the flame be extinguished when a length l of candle remains, the candle will rise out of the water if v be $> \sqrt{\dfrac{\sigma l g}{\rho}}$; but if v be $< \sqrt{\dfrac{\sigma l g}{\rho}}$ the time of an oscillation will be $= 2\pi\sqrt{\dfrac{\rho l}{\sigma g}}$.

38. A right cone is floating with its axis vertical and vertex downwards in a fluid, and $\dfrac{1}{n}$th part of the axis is immersed; a weight equal to the weight of the cone is placed on the base, upon which the cone sinks till its axis is totally immersed, before rising, shew that

$$n^3 + n^2 + n = 7.$$

39. A cup whose outside surface is a paraboloid of revolution of latus rectum l, and whose thickness measured horizontally is the same at every point and very small compared with l, has a circular rim at a height h above the vertex, and rests on the highest point of a sphere of radius r. If water be now poured in until its surface cuts the axis of the cup at a distance $\dfrac{3}{20}h$ from the vertex, and if the weight of water be four times that of the cup, shew that the equilibrium will be stable, if

$$\frac{h}{l} < \frac{r - 2l}{2r + l}.$$

40. An isosceles triangular lamina ACB is at rest with its plane vertical, and its vertex C fixed at a depth c below the surface of a liquid, the density of which varies as the depth. If the density of the lamina be the same as that of the liquid at the depth d, and if θ be the angle which the altitude h of the triangle makes with the vertical, prove that

$$8dh^3 \cos^2\overline{\theta + \alpha} \cdot \cos^2\overline{\theta - \alpha} = 3c^4 \cos^2\alpha \cdot \cos\theta,$$

the angle ACB being 2α.

41. If a solid of revolution be immersed in a heavy homogeneous fluid with its axis vertical, prove that, when the total normal pres-

sure on the surface is a minimum, its form must be such that the numerical value of the diameter of curvature of the meridian at any point is a harmonic mean between the segments of the normal to the surface at that point intercepted between the point and the surface of the fluid and between the point and the axis, respectively.

42. A hollow cylinder of height $2h$ and radius c with both ends closed contains water, and is placed with the centre of its base in contact with the highest point of a rough sphere of radius r; the weight of the water is equal to that of the cylinder, shew that the equilibrium will be stable if the water occupy a length of the cylinder which lies between the roots of

$$2x^2 - 4(2r - h)x + c^2 = 0.$$

43. A parabolic lamina, bounded by a double ordinate perpendicular to the axis, floats in a liquid with its focus in the surface and its axis inclined at an angle $\tan^{-1}\frac{\sqrt{7}}{2}$ to the vertical; prove that the density of the liquid is to that of the lamina as 216 : 121, and that the length of the bounding ordinate is three times the latus rectum.

44. A weightless shell in the form of a paraboloid of revolution rests in a similar shell, the parameter of which is double that of the former, and contains fluid whose density varies as $(\text{depth})^n$. Find the depth of the fluid in order that the equilibrium may be neutral.

45. A conical vessel of height h, vertex downwards, is filled with liquid the density of which is λx, x being the depth. This is poured into another vessel in the form of a surface of revolution, and it is found that the new density is μx^2. Prove that the form of the vessel is given by the equation,

$$y^2 + z^2 = \frac{2\mu}{\lambda} x \left(h - \frac{\mu}{\lambda} x^2\right)^2 \tan^2 \alpha.$$

46. An indefinitely small piece of ice, the shape of which may be taken to be that of a right circular cylinder, is floating with its axis vertical in water. The part immersed receives deposits of ice in such a manner as to continue cylindrical, the radius and axis receiving equal increments in equal times. Find the ultimate shape of the part not immersed.

If the specific gravity of ice be ·96, prove that the surface is formed by the revolution of $y^2(9x - y)^{25} = a^{27}$.

47. A solid is composed of two cubes, symmetrically joined together, but of different material and size. It floats with the common plane in the surface of a fluid. Find the condition of stability.

48. A small spherical cavity (radius $= R$) in an attracting mass is filled with a homogeneous incompressible fluid, and the attraction at the centre of the sphere is evanescent: prove that the fluid pressure at the centre cannot be less than $-\frac{1}{2}\rho c R^2$, and the total pressure on the surface of the cavity not less than $-\left(c + \frac{4}{3}\pi\rho\right) 2\pi\rho R^4$, where ρ is the density of the fluid, and, U denoting the potential of the attracting mass, c is the least algebraical value of $\frac{d^2U}{ds^2}$ at the centre for an element ds drawn in any direction from the centre.

49. A soap-bubble of uniform thickness is filled with a gas of such density that the weight of the whole is equal to that of the air displaced; find the form of the bubble, which is supposed to differ but little from a sphere.

50. A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid pressure. Supposing that the tensions in the meridian and circular sections are regulated by Hooke's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x + A = \int \frac{\frac{py^2}{2} + B}{\sqrt{\left(\frac{\lambda y^2}{2a} - \lambda y + C\right)^2 - \left(\frac{py^2}{2} + B\right)^2}}\, dy,$$

where a is the original radius, λ one of the moduli of elasticity, and A, B, C constants of integration.

51. An ellipsoid floats with the least axis ($2c$) vertical in a fluid of twice its density, and makes small oscillations in a vertical plane about a point in the major axis ($2a$) which is fixed. Shew that the period is

$$2\pi \sqrt{\frac{8}{15} \frac{c}{g} \frac{5\kappa^2 + a^2 + c^2}{\kappa^2 + ac}},$$

where κ is the central distance of the fixed point.

52. A thin film of fluid is made to rotate about an axis with constant angular velocity, so as to form a surface of revolution. Shew that if gravity be neglected the meridian curve will be such that

$$\frac{1}{\rho}+\frac{1}{\rho'}=\frac{x^2}{a^3}+\frac{1}{b}.$$

53. A cylindrical aperture cut through a solid cylinder, with its axis parallel to that of the cylinder, is filled with fluid of the same density as the cylinder, and closed so that no fluid can escape; the cylinder being made to roll on a rough horizontal plane, shew that its motion will be uniform.

Determine also the surfaces of equal pressure at any instant, and trace their changes through a whole revolution of the cylinder.

54. A slender fluid ring revolves uniformly round a centre of force situated at its centre, the force varying inversely as the square of the distance; find approximately the form of a section of the ring.

55. A vessel of given capacity, in the form of a surface of revolution with two circular ends, is just filled with inelastic fluid which revolves about the axis of the vessel, and is supposed to be free from the action of gravity: investigate the form of the vessel that the whole pressure which the fluid exerts upon it may be the least possible, the magnitudes of the circular ends being given.

Shew that, for a certain relation between the radii of the circular ends, the generating curve of the surface is the common catenary.

56. A vertical tube, open at both ends and of the same transverse section throughout, is kept at a uniform temperature: supposing the increase of temperature of any portion of air within the tube to be proportional to the time, shew that the velocity of the current of air at a distance x from the bottom is given by the equation

$$\frac{v^2=2gx}{2gc_1}=\log\frac{gv+ka}{c_2}+\frac{ka}{gv+ka}.$$

How may the constants c_1, c_2 be determined?

57. A small body floats on the surface of a liquid which rotates with uniform angular velocity about a vertical axis: shew that it

cannot be in relative equilibrium unless its centre of gravity either lies on the axis or coincides with the centre of gravity of the fluid displaced, and find what will happen when neither of these conditions is fulfilled.

58. If there be vibrating plates at each end of a tube, how must the times of vibration be related, so that musical notes may be produced?

59. The same musical note is produced in each of two railway trains travelling in the same direction; shew that a person standing on the line between the trains will hear beats, and determine the number of beats per second, when the pitch of the note and the velocities of the trains are given.

60. A uniform bent tube of given length, the two legs of which are vertical and of the same height, is filled with fluid. A heavy plug exactly fitting the tube is placed upon the surface of the fluid in one of the open ends of the tube, and is allowed to descend by its own weight; determine the greatest depth to which it will sink; and if the length and weight of the plug be small, shew that it will displace very nearly twice its own weight of the fluid.

In the latter case determine the amplitude and time of an oscillation.

61. A spherical homogeneous solid earth, supposed to be fixed, is surrounded by a shallow sea, which is attracted by a distant fixed body; prove that, neglecting the attraction of water on itself, the surface of the sea will remain spherical, but that its centre will deviate from the centre of the earth by a distance amounting to the same fraction of its radius that the attraction of the disturbing body is of the attraction of the earth on an element of the liquid.

62. An isosceles triangular lamina, of which the sides AB, AC are equal, floats with the angular point downwards in a liquid of which the density varies as the depth: if AD be perpendicular to BC, prove that if the lamina can float with the line AD inclined at an angle θ to the vertical, θ is given by the equation,

$$81\sigma \sin^3\theta = 64\rho \cos^8 \alpha\, (\sin^2\theta - \sin^2\alpha)^2,$$

where 2α is the angle BAC', σ is the density of the lamina, and ρ is the density of the liquid at a depth equal to AB or AC.

63. A portion of a paraboloid, latus rectum $4a$, is cut off by a plane perpendicular to the axis at a distance $3a$ from the vertex; if the vertex of the paraboloid be fixed at a depth $\frac{\sqrt{3}}{2}a$ beneath the surface of a liquid, shew that it will rest with the focus in the surface if the ratio of the density of the liquid to that of the solid be 729 : 232.

64. A solid paraboloid of height h and latus rectum $4a$, is in equilibrium in a vertical position, with its vertex downwards, and is moveable about its vertex, which is fixed at a given depth c below the surface of a liquid, the density of which varies as the depth; prove that the equilibrium is stable if the ratio of the density of the paraboloid to the density of the liquid at the depth of its vertex is less than the ratio of $c^3 + 4ac^2$ to $4h^3$.

65. A weightless piston exactly fits a hollow vertical cylinder of length l open at the top, and connected from below with a steam-boiler. The upper end of the piston resting at the bottom of the cylinder, fluid of specific gravity σ is poured in to a depth c. Steam of pressure p is now admitted, and communication with the boiler is cut off when the piston has ascended through a space a $(a + c < l)$. Shew that the piston will be reduced to rest when it has described a space x given by the equation

$$\log\frac{xc}{(l-c)(l-x)} = \frac{\sigma l}{pa}gx - \frac{l}{c}\left(\log\frac{l-c}{a} + 1\right).$$

66. A liquid flows with a known varying velocity through a tube AB of small uniform section, the axis of the tube being a plane curve and its end B free: a certain point O, determined at each instant by the state of motion of the fluid, is taken on the tangent at B, and Q is the position assumed by O when the tangent has rolled on the curve till its point of contact is P, and OY is the perpendicular from O on the tangent at P: prove that the longitudinal tension at P is proportional to QY, the transverse stress to OY, and the bending moment to the area BOP.

67. A uniform wedge, whose section perpendicular to its edge is an isosceles triangle of which the semi-vertical angle is $\tan^{-1}\sqrt{2}$ and base b, floats with its edge fixed in the surface of a liquid of twice its specific gravity; prove that, if it be depressed through a small angle β

about the vertex, the time in which it will return to its original position is approximately

$$\frac{1}{8\beta}\sqrt{\frac{5b}{\pi g}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2.$$

68. An infinite mass of liquid, not acted upon by any forces, is at rest, the pressure at infinity being given.

If the portion contained within the surface,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

be annihilated, prove that the pressure at the point ξ, η, ζ, is diminished in the ratio

$$\int_0^{\mu_1}\frac{d\lambda}{P}:\int_0^{\infty}\frac{d\lambda}{P},$$

where $P^2=(a^2+\lambda)(b^2+\mu)(c^2+\nu)$, and μ_1 is the positive root of the equation,

$$\frac{\xi^2}{a^2+\mu}+\frac{\eta^2}{b^2+\mu}+\frac{\zeta^2}{c^2+\mu}=1.$$

69. Fluid is moving uniformly in a capillary tube under a constant pressure ϖ in the direction of its length. Supposing the tube to be of constant bore and that the fluid friction varies as the relative velocity, then the velocity at any point will be given by

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{\varpi}{\mu l}=0.$$

If the tube be circular and if the film in contact with the tube be supposed to be at rest; then the velocity at a distance (r) from the axis is

$$\frac{\varpi}{4\mu l}\cdot(a^2-r^2),$$

a being the radius of the tube and μ the coefficient of resistance.

70. A homogeneous liquid is contained between two concentric spherical surfaces, the radius of the inner being a and that of the outer indefinitely great. The fluid is attracted to the centre of these surfaces by a force $\phi(r)$, and a constant pressure Π is exerted at the outer surface.

Suppose $\int\phi(r)\,dr=\psi(r)$, and that $\psi(r)$ vanishes when r is infinite.

Shew that if the inner surface is suddenly removed, the pressure at the distance r is suddenly diminished by

$$\Pi \frac{a}{r} - \frac{a\rho}{r} \psi(a).$$

Find $\phi(r)$ so that the pressure immediately after the inner surface is removed may be the same as it would be if no attractive force existed. Also with this value of $\phi(r)$, find the velocity of the inner boundary of the fluid at any period of the motion.

71. In the steady motion of a liquid in two dimensions the stream lines are the lines of constant angular velocity.

72. A spherical shell of homogeneous gravitating liquid, having no initial motion, is left to itself; find the pressure at any point during the collapse.

73. If the earth be supposed spherical and covered with an ocean of small depth, and if the attraction of the particles of water on each other be omitted, the ellipticity of the ocean spheroid will be given by the equation,

$$2\epsilon = \frac{\text{centrifugal force at the equator}}{\text{force of gravity at the earth's surface}}.$$

74. A pneumatic railway carriage can move freely without friction in a tunnel which it exactly fits. It is placed at rest at one end, and an engine begins to exhaust the air at the other, pumping out equal volumes in equal times.

Shew that at time t the distance of the carriage from the end to which it is travelling is determined by an equation of the form

$$x \frac{d^2x}{dt^2} + b \frac{dx}{dt} + n(x + bt) = na.$$

75. Elastic fluid is moving in a straight tube and is not acted upon by any forces; if the velocity at any point be independent of the time, prove that it is given by an equation of the form,

$$v^3 - 3kv = a^2x + b^3.$$

76. A globule of inelastic fluid falls under gravity through a medium, which exerts on it at any point of its surface a pressure which is equal to a constant pressure increased, if the surface at that point be moving in the direction opposite to that in which the pres-

sure acts, and diminished, if in the same direction, by a quantity proportional to the normal velocity of the surface at that point; shew that when the globule has attained its terminal velocity its figure will be a sphere.

77. At a station on a railway passed at full speed by a train, a certain musical note is sounded; explain the difference of the sounds heard by a person in the train as it approaches to and recedes from the station.

78. If a body float at rest, shew that for any displacement, consistent with the condition that the weight of the fluid displaced be equal to that of the float, the difference of the distances of the centres of gravity of the float and of the fluid displaced below the surface of the fluid will, in general, be a maximum or minimum according as the equilibrium is unstable or stable.

Moreover if Z be this difference, and the body be symmetrical with respect to a vertical plane, perpendicular to the line about which the displacement aforesaid is made, and θ be the inclination of any fixed line in the body and in that plane to the vertical, the time of a small oscillation will be that of a simple pendulum of which the length is $\dfrac{k^2}{\frac{d^2Z}{d\theta^2}}$, where k is the radius of gyration about a line through the centre of gravity parallel to the axis of displacement.

Mention any conditions which limit the generality of these theorems.

79. The particles of a fluid, following the law "pressure varies as the square of the density," and subject to no external forces, are in motion in a cylindrical tube. If x be the distance of a transverse section from a fixed point in the tube, and v be the velocity of the particles of that section at time t, prove the equation

$$\frac{d^2v}{dt^2} + 3\frac{dv}{dx}\left(\frac{dv}{dt} + v\frac{dv}{dx}\right) + 2v\frac{d^2v}{dxdt} + \frac{d^2v}{dx^2}\left(\frac{3v^2}{2} + \int_0^x \frac{dv}{dt}\,dx\right) = 0.$$

80. A portion of homogeneous fluid is confined between two concentric spheres radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance, the inner spherical surface is suddenly annihilated, and when the radii of the inner and outer surfaces of the fluid are r and R, the fluid

impinges on a solid ball concentric with their surfaces, prove that the impulsive pressure at any point of the ball for different values of R and r varies as

$$\sqrt{(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r} - \frac{1}{R}\right)}.$$

81. A solid body is floating in a liquid of variable density and its position is slightly changed so that the mass of liquid displaced remains unaltered. If $f(z)$ be the density at a depth z, and (x, y, z) the co-ordinates of any point in the immersed surface of the body, referred to the surface as the plane xy, prove that the point in the plane of floatation about which the body turns is the centre of gravity of that plane treated as a lamina, the density of which at the point (x, y) is $f(z)$.

82. Prove that in order that indefinite plane waves may be transmitted, without alteration, with a uniform velocity a in a homogeneous fluid medium, the pressure and density must be connected by the equation

$$p - p_0 = a^2 \rho_0^2 \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right),$$

where p_0 and ρ_0 are the pressure and density in the undisturbed part of the fluid.

83. Air is confined between two planes near each other and a small circular disturbance is excited about any point; investigate the differential equation for the propagation of the motion, and prove that, if $\frac{d\phi}{dr}$ be the velocity of a particle at a distance r from the origin, the equation is satisfied by

$$\phi = \int_0^\pi \psi(at + r\cos\theta)\, d\theta$$

where ψ is an arbitrary function.

84. Prove that, if the Earth be considered as a homogeneous mass of fluid in the form of a spheroid, revolving with a uniform angular velocity about its axis, gravity at any point acts along the normal, and is proportional to the part of the normal intercepted between the point of contact and the plane of the equator.

If the Earth be completely covered by a sea of small depth, prove that the depth in latitude l is very nearly equal to $H(1 - \epsilon\sin^2 l)$, where H is the depth at the equator, and ϵ the ellipticity of the Earth.

85. A mass (M) of homogeneous liquid revolves in relative equilibrium about a fixed axis with a uniform angular velocity such that the ellipticity (ϵ) of its surface is small. If the part μM of the mass were collected into an infinitely dense material point at the centre, and the density of the remaining part $(1-\mu) M$ were diminished in the ratio of $1-\mu$ to 1, find what would be the ellipticity of the new surface of equilibrium, supposing the time of rotation to be the same as before.

86. A hollow sphere, of radius b, contains within it a concentric sphere of radius a, the space between being filled with water at rest. If the inner sphere begin to move with velocity V, prove that the initial radial and transversal velocities of the water at the distance r from the centre are

$$\frac{a^3(b^3-r^3)}{r^3(b^3-a^3)} V\cos\theta \text{ and } \frac{a^3(b^3+2r^3)}{2r^3(b^3-a^3)} V\sin\theta.$$

87. A hollow ellipsoid is filled with liquid originally at rest and is set in motion about its centre of inertia; prove that the velocity-function of the liquid is

$$\phi = \omega_1 \frac{b^2-c^2}{b^2+c^2} yz + \omega_2 \frac{c^2-a^2}{c^2+a^2} zx + \omega_3 \frac{a^2-b^2}{a^2+b^2} xy,$$

where ω_1, ω_2, ω_3 are the components of the velocity of rotation of the ellipsoid, and x, y, z coordinates relative to the axes of the ellipsoid.

Shew also that the components of the velocity of the liquid relative to the ellipsoid are

$$\frac{dx}{dt} = \frac{2a^2\omega_3 y}{a^2+b^2} - \frac{2a^2\omega_2 z}{a^2+c^2},$$

$$\frac{dy}{dt} = \frac{2b^2\omega_1 z}{b^2+c^2} - \frac{2b^2\omega_3 x}{b^2+a^2},$$

$$\frac{dz}{dt} = \frac{2c^2\omega_2 x}{c^2+a^2} - \frac{2c^2\omega_1 y}{c^2+b^2}.$$

And that if the ellipsoid revolves about a fixed axis, after

$$\left\{\left(\omega_1 \frac{2bc}{b^2+c^2}\right)^2 + \left(\omega_2 \frac{2ca}{c^2+a^2}\right)^2 + \left(\omega_3 \frac{2ab}{a^2+b^2}\right)^2\right\}^{-\frac{1}{2}}$$

revolutions of the ellipsoid, every particle of the liquid will be in the same position relative to the ellipsoid.

88. If a thin ellipsoidal shell without mass be filled with water, and set in motion about its centre as a fixed point, prove that its

subsequent motion will be determined by three equations of the form

$$\frac{(b^2-c^2)^2}{b^2+c^2}\frac{d\omega_1}{dt}+\frac{(b^2-c^2)(b^2c^2+c^2a^2+a^2b^2-3a^4)}{(c^2+a^2)(a^2+b^2)}\omega_2\omega_3=L.$$

89. The transverse section of a uniform prismatic closed vessel is of the form bounded by the two intersecting hyperbolas,

$$\sqrt{2}\,(x^2-y^2)+x^2+y^2=a^2,\qquad \sqrt{2}\,(y^2-x^2)+x^2+y^2=b^2.$$

If the vessel be filled with water, and be made to rotate with angular velocity ω about its axis, prove that the initial component velocities of any point (x, y) of the water will be

$$\frac{\omega}{a^2+b^2}\{2y^3-6x^2y+\sqrt{2}\,(a^2-b^2)\,y\},\text{ and }\frac{-\omega}{a^2+b^2}\{2x^3-6xy^2+\sqrt{2}\,(b^2-a^2)x\}.$$

90. In a uniform tube of indefinite length is placed a disc which fills it and makes n complete revolutions in a second, their amplitude being c; another disc of mass M is placed at a distance l from the first, and is supported by a spring, whose elasticity is such that the disc, if vibrating freely, would make m vibrations in a second; shew that after a sufficient time has elapsed for the excursions of the air in the tube beyond the second disc to become uniform their amplitude will be

$$c'=c\cos\beta\,\{1-2\sin\beta\sin\gamma+\sin^2\beta\}^{-\frac{1}{2}},$$

where $$\tan\beta=\pi\frac{Mn}{\rho v}\left(\frac{m^2}{n^2}-1\right),\text{ and }\gamma=\beta+4\pi\frac{ln}{v},$$

ρ being the density of air, and v the velocity of sound.

Find the values of l for which c' is a maximum or a minimum, and shew that the maxima are greater and the minima smaller the greater the value of $\tan\beta$.

91. A closed vessel is filled with homogeneous liquid and moved in any manner about a fixed point O. If at any time the liquid were removed, and a pressure proportional to the velocity potential were applied at every point of the surface, the resultant couple at O due to the pressures would be of magnitude G, and its axis would be in a line OQ. Shew that the energy of the fluid was $\frac{\rho\omega G}{2\mu}\cos\theta$, ω being the angular velocity of the surface, θ the angle between the direction of ω and OQ, ρ the density, and $\mu\phi$ the pressure at a point where ϕ was the velocity potential.

92. A hollow prism whose edges are vertical and ends horizontal has for a horizontal section the equilateral triangle ABC, of which

D, E, F are the middle points of the sides, G the centre of gravity, and K, L, M the middle points of AG, BG, CG. The prism is filled with water, and the whole is set rotating about a vertical axis through D. Shew that if the prism be suddenly stopped, the fluid will be initially at rest at A, B, C, G; that the initial velocities at D, E, F are equal, and also those at K, L, M; and that the velocity at D is three times the velocity at K, and in the opposite direction.

93. The diameters of the strings of a violin, supposed to be of the same material, are as 2, 3, 4, 6. The velocity of transmission of the transversal vibrations on any string is to that on the string of next higher note as 2 to 3. Find the position of the sounding-point, supposed to be placed in the direction of the resultant of the pressures on the bridge of the violin, the curvature of which is neglected.

94. A small rigid vertical cylinder, containing air, is rigidly closed at the bottom, and covered at the top by a disc of very small weight which fits it air-tight. Supposing the air in the cylinder to be set in vibration, prove that the period of a vibration is $\frac{2\pi}{m}$, m being a root of the equation
$$m \tan \frac{ml}{a} = \frac{\kappa\beta\Pi}{\mu a};$$
where l is the length of the tube, a the velocity of sound in air, μ the mass, κ the area of the disc, $p \propto \rho\,(1 + \beta s)$ the relation between the pressure and density when the latter is suddenly changed from ρ to $\rho\,(1 + s)$, and Π the pressure of the air in the cylinder before motion commences.

95. If a stretched string of length l be set in vibration, and if its initial form be $y = c \cos \frac{\pi x}{l}$, prove that after the time t its form is
$$y = \frac{c}{\pi} \Sigma_1^\infty \frac{8n}{4n^2 - 1} \sin \frac{2n\pi x}{l} \cos \frac{2n\pi at}{l}.$$

96. A string is stretched between two points distant l and then displaced into the curve
$$\frac{y}{k} = e^{-\frac{x}{l}} . \sin \frac{\pi x}{l},$$
shew that its form at any subsequent time t is
$$\frac{y}{k} = \Sigma_{n=1}^{n=\infty} \frac{4n\pi^2 (1 + e^{-1} \cos n\pi) \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}}{\{1 + (n-1)^2 \pi^2\}\{1 + (n+1)^2 \pi^2\}},$$
a being the constant velocity of propagation of waves along the string.

97. The portions Ax, Ax' of a stretched string xAx' are of different thicknesses; prove that any small transversal vibrations travelling from x to A will, on arriving at A, be partly reflected and partly transmitted to Ax', and that the displacements due to the incident, reflected and transmitted vibrations are to each other as $1+\mu : 1-\mu : 2$, where μ is the ratio of the velocity of propagation in Ax to that in Ax'.

98. A cord is stretched between two points, and loaded at its middle point with a small weight w; shew that, if t be the time of vibration of the cord for any possible note, the values of t are given by the equation

$$\tan\left\{\frac{\pi l}{t}\sqrt{\frac{m}{\tau}}\right\} = \frac{t}{\pi w}\sqrt{m\tau},$$

where l is the length, τ the tension, and m the mass of a unit of length of the cord.

99. Supposing the effect of friction in the case of aerial vibrations in a tube of uniform bore to be the production of a retarding acceleration on each particle equal to $f\times$ velocity, shew that the equation of motion will be satisfied by taking as the type of the vibrations the expression

$$C\epsilon^{-\frac{1}{2}ft}\sin\frac{2\pi}{\lambda}\left\{at\sqrt{1-\frac{f^2\lambda^2}{16\pi^2a^2}}-x\right\},$$

where a is the velocity of propagation.

100. In the transverse vibrations of an elastic string, if a retarding force be supposed to act on the elements of the string proportional to the velocity, obtain the equation

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}=a^2\frac{d^2y}{dx^2}.$$

Shew that if the string be of length l, and if the ends be fixed, a general solution of this equation is

$$y=e^{-kt}\Sigma_{i=1}^{i=\infty}A_i\sin\frac{i\pi x}{l}\sin\left\{\left(\frac{i^2\pi^2a^2}{l^2}-k^2\right)^{\frac{1}{2}}t+\beta_i\right\}.$$

Determine A_i and β_i in the special case, in which the string was pulled aside to a distance b, at a point distant c from one end, and then let go.

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